Challenge and Thrill of Pre-College Mathematics

New Age International Publishers

NEW AGE

SECOND EDITION

Challenge and Thrill of Pre-College Mathematics

V Krishnamurthy
C R Pranesachar
K N Ranganathan
B J Venkatachala

NEW AGE INTERNATIONAL PUBLISHERS

V Krishnamurthy

C R Pranesachar

K N Ranganathan

B J Venkatachala

Challenge and Thrill of Pre-College Mathematics

(SECOND EDITION)

(In two colour)

V Krishnamurthy
Former Professor
Department of Mathematics and
Director, K K Birla Academy
New Delhi

C R Pranesachar
Former Professor
SDGS College, Hindupur
Andhra Pradesh

K N Ranganathan
Former Professor
Department of Mathematics
Rama Krishna Mission Vivekananda College
Mylapore, Madras

B J Venkatachala
Former Professor
Kuvempu University
Karnataka

Publishing Globally

NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS
New Delhi • Bangalore • Chennai • Cochin • Guwahati
Hyderabad • Kolkata • Lucknow • Mumbai
Visit us at www.newagepublishers.com

PREFACE TO THE FIRST EDITION

The germination of this book goes back to the experience of the National Board for Higher Mathematics (NBHM) in conducting the training programmes for the successive International Mathematical Olympiads in which India has been participating since 1989. June 1990 saw the birth of the Project for writing an enrichment text for nurturing mathematical talent in the country. This book is one of the outcomes of this effort.

The book is intended for students of the ninth, tenth and eleventh standards, especially for the top half or the more gifted of the population. It is to be studied mostly by the student on his own, parallel to or in advance of the routine coverage of the subject in school. In one sense it is self-contained since it treats every topic from scratch. But it quickly ventures into concepts, ideas, proofs and problems which the school syllabus usually shies away from, not because they are above the syllabus but because the audience aimed at is the whole population. Catering to the large spectrum that ranges from first generation learners to the talented ones, the school treatment necessarily errs by staying at the average level. We concur with the view of Jamshedji Tata, the farsighted pioneer and visionary of scientific research and development in India, when he says that

> "*What advances a natian or community is not so much to prop up its weakest and most helpless members as to lift up the best and most gifted so as to make them of the greatest service to the country. I prefer this constructive philanthropy which seeks to educate and develop the faculties of the best of our young men*".

As such the book aims high at the larger objective of motivating the student to recognise and enjoy the pleasures of a mathematical pursuit. It emphasises principles and exploits the challenge of problem-solving. It makes the fundamentals of mathematics secure for the student so that he does not have to unlearn anything when he reaches the tertiary level of education. The purpose is not to stuff the student with more bare matter, but to broaden the base so that in due time he can see deeper tilings more comfortably. An attempt has therefore been made to avoid the mystification of doling out forbidding formulae without even a heuristic justification, at the same time also avoiding a fetish of rigour which comes in the way of effective communication. The evolution of general concepts is done from concrete examples and special cases, thereby making the transition to abstraction smooth and natural. The mutual enrichment of abstraction and concretion is a dominant part of the culture of mathematics. The younger the student is while he is first introduced to this culture, the more convincing will be his acceptance of the logical nuances of Higher Mathematics when he enters

that royal mansion. Naturally we do not hesitate to use ideas from different branches of Mathematics. The consequent cross-fertilisation and the habit of looking at Mathematics as an integrated discipline result as every Mathematician knows, in an enjoyable perception of Mathematics.

Except for Calculus and Statistics — which two topics the book does not touch — it starts from the ninth standard level but reaches even *far beyond* the twelfth standard level in its sophistication under each topic — thus bridging a much-lamented gap between school mathematics and university mathematics. Not less than 300 problems have been worked out with detailed explanations regarding strategy, modelling, manipulation, abstraction and notation. Accordingly, the active participation of the learner is required in the understanding of the book — which, we hope, is assured, because of the nature of the clientele for which it has been written. In some chapters where the learning of mathematics is more by ideation than a routine exposure to drill problems, we have ventured to save space for the more creative type of problems. It is hoped that those who will use this book intensively on their own will be the large majority of higher secondary level students who want to relearn their mathematics in order to develop a stronger interest and a better appreciation of what they are already expected to know. Actually these students are advised to keep this book with them as a constant companion throughout their higher studies at college level. In fact, the book can be effectively used by all long-distance learners and students of the various wings of non-formal education in the country.

We are extremely grateful to the NBHM and to its chairman Prof M.S. Raghunathan for suggesting this book-writing project and for their sustained support in carrying out the project. But we the authors, hold the academic responsibility for the book and, so, for all the errors in the book.

Acknowledgements and appreciation are due to

(*i*) several colleagues for various constructive suggestions;

(*ii*) Students' Xerox, Madras, for the first typing of the entire script;

(*iii*) New Age International Ltd., for the publication;

(*iv*) Mrs. Dr. Usha Ranganathan, IIT Madras and Ms. Radha, K.K. Birla Academy, New Delhi for help in proof correction and preparation of answers; and

(*v*) Professor Phoolan Prasad, Member, NBHM, for his valuable encouragement throughout.

We shall welcome all suggestions for the improvement of the book.

V. KRISHNAMURTHY

C.R. PRANESACHAR

K.N. RANGANATHAN

B.J. VENKATACHALA

A Prologue for the Student–Reader

Mathematics is essentially an abstracting science of concrete situations. Today it is the only language of precise communication and technical understanding of almost every field of human endeavour. At the first dawn of civilization, it all started with numbers and forms. The study of the former gave rise to arithmetic and algebra; the study of the latter initiated the discipline of geometry. In this book we shall study these basic branches of mathematics in such a way that what we study would become a sure and strong foundation for everything that follows in Mathematics at higher levels. In this sense, for the student, the topics covered in this book would be the starting point of a life long venture into the labyrinths of Higher Mathematics, ultimately leading him to the comprehension and application of the subtlety, the beauty and the culture of a mathematical way of thinking.

Reader, a brief bird's eye-view of what is in store for you as a student, would be in order here. In the first chapter you will be introduced to the different classes of numbers with which you are probably intuitively familiar. During the long history of Mathematics, these number systems together constituted the motivation and the spring board for mathematicians, for diving deep into the ocean of mathematics. Chapter 2 gives the elementary properties of integers with plenty of illustrations and applications. Chapters 3 and 4 tell you elaborately everything that is basic in the geometry of straight lines and circles. Here you will have a refreshing training in the tight-rope walking on the delicate bridges of mathematical logic – that gives mathematics its unique culture of a precise reasoning from cause to effect. You will meet with several illuminating problems and geometric constructions in this chapter. Chapter 5 begins the study of algebraic equations of the second degree and thereby introduces you to geometric curves other than the circle.

Chapter 6 opens up a new vista of mathematics by introducing you to what are called Trigonometrical ratios. In every branch of scientific study, you would find that these trigonometrical ratios (or circular functions, as they are also called) would be as common in your investigations as numbers and arithmetic are in one's daily life. Chapter 7 connects algebra and geometry in an ingenious way which has created history. Here was born what may be called modern mathematics. You are introduced for the first time to the strategy of using algebraic methods in geometry. In fact geometry transforms into algebra by these methods. From here onwards there should be no hesitation for you to look at Mathematics as an integrated discipline – not as arithmetic, algebra and geometry separately. In Chapter 8 this strategy is effectively used by investigating a

system of several equations of the first degree, using the convenience of a geometrical visualisation. By this time, student reader, you would see how powerful mathematical methods could be and here in this chapter you would get a glimpse of why abstraction in mathematics is so effective an instrument in the hands of the modern user of mathematics. Chapter 9 introduces you to certain intricacies in the very counting process, which one has learnt even from childhood. You are advised not to under-estimate the mathematics of this chapter; for, the 'simplicity' of the counting process can be very deceptive. If you master the methods of this chapter, you would discover that your power to count has increased multifold.

Chapter 10 now takes you to the next higher step in Mathematics by showing how, instead of just numbers and symbols, you can now play and manipulate with 'polynomials'. This introduction would be your grand entry into the technical mansion of mathematics. It is recommended that you spend a considerable time in mastering the foundation laid in this chapter - because here for the first time you are led into the subject of Higher Algebra proper. Chapter 11 familiarizes you with a major technique of mathematical manipulation and estimation, namely, Inequalities.

Chapter 12, 13 and 14 are the bare beginnings of three important branches of Mathematics:- the first, being a foundation for one of the most applicable areas called Combinatorics; and the second, being the starting point of a vast area of mathematics called Probability Theory that has penetrated into every science not merely as a tool but as the only operational way of handling the particular topic of that science. Chapter 14 lays the foundation for Number Theory which contains several unsurpassed gems in Mathematics. Chapter 15 serves as a connecting link between the finite operations of Mathematics dealt with in this book and the infinite operations that are a characteristic of a large part of advanced mathematics, that follows at University level. In conclusion, Chapter 16 opens the door for the fascinating algebra of Complex Numbers.

Dear Reader, throughout, you will find that mathematics has a special purpose, namely, it solves problems. In equipping you for problem solving it takes you into the most original creations of the human mind, together called, Mathematics, the kingpin of scientific thinking.

Authors

Contents

CONTENTS

Publishers' Note (v)

Preface to the First Edition (vii)

A Prologue for the Student-Reader (ix)

1. Number Systems **1**

2. Arithmetic of Integers **14**

2.1 The Principle of Induction 14

2.2 Divisibility 18

2.3 Fundamental Theorem of Arithmetic 27

Problems 34

3. Geometry—Straight Lines and Triangles **36**

3.1 Straight Lines 36

3.2 Congruence of Triangles 42

3.3 Parallel Straight Lines 50

3.4 Some Properties of a Triangle 58

3.5 Similar Triangles 61

3.6 Congruence and Collinearity 74

Problems 81

4. Geometry-Circles **86**

4.1 Circles-Preliminaries 86

4.2 Tangents 91

4.3 Cyclic Quadrilaterals 99

4.4 Triangles Revisited 105

4.5 Constructions 132

4.6 Some Geometric Gems 155

Problems 171

5. Quadratic Equations and Expressions **177**

5.1 Introduction 177
5.2 Solution of Quadratic Equations by Factorization 181
5.3 Method of Completing the Square 183
5.4 Relation Between Roots and Coefficients 188
5.5 Problems Leading to Quadratic Equations 192
5.6 Behaviour of Quadratic Expressions 194
Problems 199

6. Trigonometry **202**

6.1 Introduction 202
6.2 Trigonometric Functions or Ratios 203
6.3 Trigonometrical Ratios of $90° \pm \theta$, etc. 206
6.4 Functions and Their Graphs 209
6.5 A : Ratios of Compound Angles 217
6.5 B : Conversion Formulae (Products into Sums) 218
6.5 C : Conversion Formulae (Sums into Products) 219
6.5 D : Ratios of Multiple Angles 220
6.5 E : Ratios of 18° and 36° 222
6.6 Trigonometrical Identities 226
6.7 Inverse Circular Functions 229
6.8 Trigonometrical Equations 234
6.9 Properties of Triangles 240
6.10 Heights and Distances 258
6.11 Elimination 262
Problems 265

7. Co-ordinate Geometry of Straight Lines and Circles **267**

7.1 Introduction 267
7.2 Straight Lines 277
7.3 Circles 302
Problems 314

8. Systems of Linear Equations **317**

8.1 Two and Three Unknowns 317
8.2 Introduction to Determinants and Matrices 328
Problems 351

9. Permutations and Combinations **354**

9.1 Permutations 355

	9.2 Combinations	367
	9.3 Binomial Theorem	382
	Problems	387
10.	**Factorisation of Polynomials**	**389**
	10.1 Introduction	389
	10.2 Addition and Multiplication of Polynomials	390
	10.3 Division of Polynomials	392
	10.4 Remainder Theorem and Factorisation	398
	10.5 GCD and LCM of Polynomials	404
	10.6 Euclid's Algorithm	407
	Problems	414
11.	**Inequalities**	**418**
	11.1 Introduction	418
	11.2 Some Basic Inequalities	418
	11.3 AM-GM Inequality	425
	11.4 Cauchy-Schwarz Inequality	429
	Problems	433
12.	**Elementary Combinatorics**	**436**
	12.1 The Inclusion and Exclusion Principle (IEP)	436
	12.2 The Pigeon-hole Principle (PHP)	445
	Problems	454
13.	**Beginnings of Probability Theory**	**456**
	Problems	470
14.	**Beginnings of Number Theory**	**472**
	14.1 Congruences	472
	14.2 Theorems of Fermat and Wilson	476
	Problems	481
15.	**Finite Series**	**484**
	15.1 Introduction	484
	15.2 Sum of an Arithmetic Progression	488
	15.3 Sum of a Geometric Progression	491
	15.4 Some Special Finite Series	492
	15.5 Summation of Finite Trigonometrical Series	498
	15.6 Summation Involving Binomial Coefficients	500
	Problems	506

16. De Moivre's Theorem and Its Applications **509**

16.1 De Moivre's Theorem 509

16.2 n^{th} Roots of a Complex Number 513

Problems 516

Miscellaneous Problems **518**

ANSWERS (to selected Questions) **529**

INDEX **543**

-

Chapter 1 Number Systems Page 1

CHAPTER

1 NUMBER SYSTEMS : N, Z, Q, R, AND C– AN OUTLINE

The *natural numbers* 1, 2, 3, 4, ... n, ... have been with each one of us since childhood. Almost all the important properties of this number set, which we shall call **N**, have been accepted by us intuitively from experience. These properties may be listed now as follows. We add a few comments where necessary.

1. The set **N** is an endless set. That is, there is no last number. The sequence of natural numbers goes on and on.
2. There is a built-in order in the set in the way we write it:

   $$1, 2, 3, 4, ..., a, ..., b, ...$$

   If b appears later in the sequence than a then b is said to be greater than a. We write this: $b > a$; or, what is the same thing, $a < b$, *i.e.*, a is less than b.
3. Every number has a successor number and, except for 1, every number has a predecessor number.
4. Any two numbers in the set can be 'added' to produce another number in the set. Recall that after one learns to count, the next thing that is learnt is to 'add'.
5. Whether one adds a to b or b to a it is the same thing—in the sense the result is the same. In other words, addition '+' is a commutative process; *i.e.*,

   $$a + b = b + a \qquad \text{for all } a, b \in \mathbf{N} \qquad (1)$$
6. Repeated addition of the same number to itself is known as 'multiplication'. Thus, for instance, 4 added to itself 5 times is nothing but 4 × 5, that is, 20.
7. This multiplication is also commutative. That is,

   $$a \times b = b \times a \qquad \text{for all } a, b \in \mathbf{N} \qquad (2)$$
8. Both the operations, addition and multiplication, have another property, called 'associativity'. This means : $a + b$ added to c and a added to $b + c$ are both the same. Symbolically,

   $$(a + b) + c = a + (b + c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (3)$$

   In the same way, we have, for multiplication,

   $$(a \times b) \times c = a \times (b \times c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (4)$$
9. Further, there is a 'compatibility' between the two processes 'addition' and 'multiplication'; namely,

   $$a \times (b + c) = (a \times b) + (a \times c)$$

   and $$(a + b) \times c = (a \times c) + (b \times c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (5)$$

This property is called 'distributivity' of multiplication with respect to addition.

These nine properties of the set **N** shall now be assumed without any further justification. Higher mathematics may require the construction of natural numbers from scratch and the derivation of these properties thereof. We do not have either the luxury of time or the necessity of logic to get into all that now, at this level.

One of the first things that we learn as we grow learning mathematics is that the system **N** of natural numbers has several deficiencies. For instance, we can solve for x, the equation: $2 + x = 3$ within the system **N**. The answer is $x = 1$. Whereas, an equation like: $3 + x = 2$ is not solvable in **N**. In other words, there is no value of x in **N** satisfying $3 + x = 2$. We know the answer is -1 but -1 is not a natural number. Thus the system **N** of natural numbers does not have solutions of the equation $a + x = b$ *i.e.*, this equation has no solution for x in **N** unless $a < b$. Mathematics develops by concerning itself with such questions and resolving the issue. In the above situation, the resolution comes like this. Mathematics invents new numbers, namely, $0, -1, -2, -3, \ldots$ expressly to satisfy the need to solve the equations $a + x = b$ even when $a \geq b$. For instance, if $a = b$, the equation is $a + x = a$. We invent the new number "0" (= zero) to be the solution of

$$a + x = a = x + a.$$

Once we include a new number "0" to the system **N** we want also to solve equations like

$$1 + x = 0; 2 + x = 0; 3 + x = 0; \ldots$$

The solutions of these are called the negatives of 1, 2, 3, ... and are written

$$-1, -2, -3, \ldots .$$

Thus the enlarged system now contains zero and all the negative integers and **N**. This new system is denoted by **Z** and is called *the set of all integers*. Thus

$$\mathbf{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

It can also be written as below, where we bring out the 'order' relation in **Z**. In other words, in the following style of listing the elements of **Z**, if a precedes b then $a < b$, or what is the same thing, $b > a$.

$$\mathbf{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

There are several points we have to note about this enlargement of **N** to **Z**. In enlarging **N** to **Z** we have been able to 'protect' or 'preserve' as many properties of **N** as possible. Precisely we mean the following:

1. **Z** is an infinite (= endless) sequence as **N** was (and is !).
2. The built-in order in **N** is still preserved. It has in fact been extended to **Z**. In other words '$a > b$' has a meaning in **Z** for every a and b in **Z** and further, if $a > b$ in **N** for two elements $a, b \in$ **N**, it is so in **Z**, even as elements of **Z**.
3. Every number in **Z** has a successor and a predecessor. Recall that in **N** the number 1 does not have a predecessor. Also any number in **N** whether considered as a member in **N** or a member in **Z** has the same successor. Similarly, any number $\neq 1$ in **N** has the same predecessor in **N** or **Z**. We express this by saying that the 'successor-predecessor' concept has been extended to **Z** without damage to the concept already existing in **N**.

4. The operation of addition already available in **N** carries over to **Z**. If $x = -a$ where $a \in$ **N**, $y = -b$ where $b \in$ **N**, we may define $x + y$ $= -(a + b)$ where + in the R.H.S. is the addition in **N**. Since $(a + b) \in$ **N**, $-(a + b) \in$ **Z**. Thus we get the familiar equality.

$$(-a) + (-b) = -(a + b)$$

Again, if $x = -a$, $a \in$ **N**, is 'added' to $c \in$ **N** we will have $x + c = (-a) + c$. This is to be taken as

$$-(a - c) \text{ if } a > c$$

and as $$c - a \text{ if } c > a \text{ or } c = a.$$

Proceeding in this way and carefully going through every new situation we get a thorough definition of addition in **Z**. We see that 'addition' is closed in **Z** — by which, we mean, two numbers in **Z** always lead to a number in **Z** by the addition process. If two numbers are already in **N** their sum is what it is in the system **N**. Thus the extension of **N** and the addition therein to **Z** has been achieved without 'damaging' the addition in **N**. This process of enlarging a number system, preserving its algebraic structure is called an *extension* of the system. Addition of zero to any number, again satisfies,

$$a + 0 = a = 0 + a \qquad \text{for all } a \in \mathbf{Z}.$$

5. Addition in **Z** continues to be commutative. In other words,

$$a + b = b + a \qquad \text{for all } a, b \in Z \qquad (1')$$

6. Multiplication in **N** can be extended to a multiplication in **Z**, without damaging the meaning of multiplication in **N** — except that, we have to make proper rules for handling the negative sign, thus: If $a, b \in$ **N**, then

$$a \times b = ab \text{ (as in } \mathbf{N})$$
$$(-a) \times (-b) = ab$$
$$(-a) \times (b) = -(ab)$$
$$a \times (-b) = -(ab).$$

Multiplication by zero however has to be controlled by a new rule, *viz.*,

$$a \times 0 = 0 = 0 \times a \qquad \text{for all } a \in \mathbf{Z}.$$

7. Multiplication in **Z** is commutative. In other words,

$$a \times b = b \times a \text{ for all } a, b \in \mathbf{Z}. \qquad (2')$$

8. The associative properties of both addition and multiplication continue to be valid in **Z**. In other words

$$a + (b + c) = (a + b) + c \qquad \text{for all } a, b, c \in \mathbf{Z} \qquad (3')$$

and $$a \times (b \times c) = (a \times b) \times c \qquad \text{for all } a, b, c \in \mathbf{Z}. \qquad (4')$$

9. The distributive property

$$a(b + c) = ab + ac$$
$$(a + b)c = ac + bc \qquad \text{for all } a, b, c \in \mathbf{Z} \qquad (5')$$

holds, as it holds in **N**.

10. Finally, we record, at one place, the special roles of the numbers 0 and 1 in **Z** as follows:

(*i*) In **Z**, 0 is the unique number which has the property:

$$0 + a = a = a + 0 \qquad \text{for all } a \in \mathbf{Z} \qquad (6')$$

(*ii*) In **Z**, 1 is the unique number which has the property:

$$1 \times a = a = a \times 1 \qquad \text{for all } a \in \mathbf{Z} \tag{7'}$$

Note that this property (*ii*) of 1 is already present in **N** for all $a \in$ **N** and it is now valid for all $a \in$ **Z**, as well.

While the number 'zero' plays a unique role as far as addition is concerned in **Z**, the number 'one' plays an exactly analogous role with respect to multiplication in **Z**. We call '0' the additive identity in **Z** and call '1' the multiplicative identity in **Z**.

If we now compare the two systems **N** and **Z** we find that **Z** is a meaningful extension of **N**. The extension protects the properties already existing in **N** as listed above. Further, **Z** has the extra property of solvability of equations of the form

$$a + x = b,\ a,\ b \in \mathbf{Z}. \tag{8'}$$

A beauty of the extension from **N** to **Z** is the following. We invented new numbers to solve $a + x = b$ with a, b, in **N**; but in the enlarged set **Z** we are able to solve $a + x = b$ for any two a, b in **Z** !

But there is one property, viz., the following, which is true in **N** but is not true in **Z**:

If $a > b$, for any three $a, b, x \in$ **N**, then $xa > xb$ (9′)

In other words, so long as we are in **N**, multiplication by the same number of both sides of an inequality preserves the same inequality. But in **Z** this property will fail for multiplication by a negative quantity. For instance, if $a < b$ in **Z** then $-a > -b$. Multiplication by a negative integer reverses the inequality.

Thus in carrying out the extension of the number system from **N** to the larger system **Z**, we could preserve many properties, we were able to solve extra equations but we had to lose something, as if we had to pay a price for the extension !

We are now going to carry this extension process through three more stages. Each time we will have the general situation (not unlike the above extension from **N** to **Z**) between the smaller and the larger systems.

(1) We preserve most of the properties of the smaller system;

(2) We achieve something extra in the larger system — something which was not available in the smaller system; and

(3) We 'pay a price' for this extension by losing some property which was present in the smaller system.

In this series of extensions there is an enormous amount of detail to be taken care of for the purpose of mathematical rigour and completion of the argument. We shall not go through all that detail. They will be duly taken up at the level of university education. Here we shall only indicate the lines of these extensions in a broad manner. The extension that we have just completed namely,

from **N** to **Z**

may be called *the first stage*. The *second stage* of the extensions is

from **Z** to **Q**

where **Q** is the set of all rational numbers. A *rational number* is a number of the form

$$\frac{p}{q},\ p,\ q \in \mathbf{Z},\ \text{with } q \neq 0. \tag{10}$$

Most often, for printing convenience, we write $\frac{p}{q}$ as p/q.

This extension is needed for solving equations of the form

$$ax = b, a, b \in \mathbf{Z}. \tag{11}$$

For instance, we cannot solve in **Z** the equation

$$2x = 1 \text{ or } 3x = -2. \tag{*}$$

We know from our knowledge of lower class arithmetic that $x = 1/2$ is the solution of $2x = 1$ and $x = (-2)/3$ is the solution of $3x = -2$. But these numbers $1/2$, $(-2)/3$ etc. are not in **Z**. In other words, neither of the equations in (*) is solvable in **Z**. In fact (11) does not have a solution in **Z** whenever a is not a factor (= divisor) of b. Therefore we create numbers like $1/2$, $(-2)/3$ ... as solutions for $2x = 1$, $3x = -2$, ... In general, we create the rational numbers (10). But once we create them we have to merge them in the additive and multiplicative structures already existing in **Z**.

First we define the equality of two rational numbers as follows.

Definition 1. Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$. We also agree to write every $n \in \mathbf{Z}$ as $\frac{n}{1}$ in **Q**.

Note. We now have the reason for writing $\frac{4}{6} = \frac{2}{3}$. In fact, always $\frac{a}{b} = \frac{pa}{pb}$ where p is any nonzero integer. Also $\frac{-5}{9} = \frac{5}{-9}$ because $(-5) \times (-9) = 45 = 5 \times 9$. So hereafter **we can safely assume that the denominators of rational numbers are positive.**

Now we define, thereby ensuring that the sum and product of two rational numbers is again a rational number,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{12}$$

and

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \tag{13}$$

It is an interesting routine to verify the following:

(*i*) The addition and multiplication defined in (12) and (13) are both meaningful *i.e.*, if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, $\frac{c_1}{d_1} = \frac{c_2}{d_2}$ then $\frac{a_1}{b_1} + \frac{c_1}{d_1} = \frac{a_2}{b_2} + \frac{c_2}{d_2}$ and $\frac{a_1}{b_1} \times \frac{c_1}{d_1} = \frac{a_2}{b_2} \times \frac{c_2}{d_2}$.

(*ii*) Addition and multiplication in **Q** are both associatitve as well as commutative.

(*iii*) The distributive property (5′) holds in **Q** as well

(*iv*) For all $a \in \mathbf{Q}$,

$$a + 0 = a = 0 + a$$
$$1 \times a = a = a \times 1.$$

All these properties are thus present both in the smaller system **Z** and in the larger system **Q**. An additional property that is present in **Q**, but not in **Z** is the solvability of equations of the form

$$ax = b, a, b \in \mathbf{Q} \text{ and } a \neq 0 \tag{14}$$

The corresponding equation (11) is not always solvable in **Z**. But (14) is always solvable in **Q**. This is the advantage of the extension from **Z** to **Q**. But there is a 'price' that we pay for this extension. Let us describe it now. If we write the numbers of **Z** in their natural order, namely,

$$\{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

we note that for every number there is a 'next greater' number. But this fails in **Q** because in **Q** there is no 'next greater' number. We shall explain what this means.

First of all note that there is a natural order in the system **Q**. It is an extension of the natural order in **Z**.

Definition 2. For any two numbers $\frac{a}{b}$ and $\frac{c}{d}$ in **Q** with a, b, c, d in **Z** and $b, d > 0$,

$$\frac{a}{b} \geq \frac{c}{d} \text{ if and only if } ad \geq bc \tag{15}$$

Illustration $\frac{2}{3} > \frac{5}{9}$, because $2 \times 9 = 18 > 15 = 3 \times 5$

$\frac{-7}{5} < \frac{-4}{3}$, because $(-7) \times 3 = -21 < -20 = (-4) \times 5$

Now let us observe what it means to say that there is no next greater number in **Q**. Consider any two rational numbers x and y with $x < y$. *Then we have* $x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2}$ $= \frac{x+y}{2} < \frac{y}{2} + \frac{y}{2} = y$. *Thus, whenever* $x < y \in Q$ *we have* $\frac{x+y}{2} \in Q$ *such that*

$$x < \frac{x+y}{2} < y.$$

This says that between any two rational numbers x and y, there is a rational number and hence *an infinity of rational numbers are there between x and y.*

In **N** as well as in **Z** this concept of 'next greater' number is valid whereas in **Q** it is not. This is the price we pay for the extra advantage we achieve in extending to **Q**, viz., the solvability of the equations of the form (14).

Now we shall proceed to *the third stage* of this series of extensions. This stage is 'from **Q** to **R**' where **R** is the set of all *real numbers*. To explain what **R** is precisely, we have to take several steps. We shall not, in this book, be able to mathematically justify all these steps. First note, the necessity for the extension arises as follows. Suppose a is a positive rational number (*i.e.*, $a \in$ **Q**, $a > 0$) but not a perfect square. Then there is no rational number x such that $x^2 = a$. For instance $x^2 = 2$ has no solution for $x \in$ **Q**. One might say what about $x = \pm\sqrt{2}$? But what exactly is $\sqrt{2}$? We prove below, in Theorem 1, that such a number $\sqrt{2}$, if it exists, cannot be in **Q**. This is the reason for looking beyond **Q**, and obtaining an extension of **Q**. In fact, by one stroke of an extension from **Q** to **R**, mathematics solves not only the problem of solutions for $x^2 = 2$ but also $x^2 = 3$ and many other such equations which are not solvable in **Q**. But first let us take up the promised theorem.

Theorem 1. There is no rational number x such that $x^2 = 2$.

The proof is based on divisibility by 2. Recall that a natural number m is even iff $m = 2p$ for some natural number p and m is odd iff $m = 2q - 1$ for some natural number q. Each natural number is either odd or even but not both; for, if $2p = 2q - 1$ then $2(q - p) = 1$. This is impossible because $q - p$ is an integer. We also note that the square of an even number is even, since

$$(2p)^2 = 4p^2 = 2 \times 2p^2$$

and the square of an odd number is odd, since

$$(2q-1)^2 = 4q^2 - 4q + 1$$
$$= 2(2q^2 - 2q + 1) - 1.$$

Now suppose there is a rational number $x = m/n$ whose square is 2. We may suppose that m and n are natural numbers such that they do not have a common factor, otherwise we may cancel the common factor and reduce the fraction. So in particular m and n cannot both be even. Since $x^2 = (m/n)^2 = 2$ we have $m^2 = 2n^2$. This means m^2 is even and so m is even *i.e.*, $m = 2p$ for some $p \in \mathbf{N}$. Then $m^2 = 4p^2$; therefore $4p^2 = 2n^2$, so that $2p^2 = n^2$. This means n^2 is even and so n is also even. Thus both m and n are even, contradicting our assumption on m and n. This proves the theorem. ❑

Let us continue our description of the extension from **Q** to **R**. A number like $\sqrt{2}$ which is not a rational number is called an *irrational number*. Thus $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots$ are all irrational numbers. A rational number is expressible as either a terminating decimal or a decimal with a recurring portion. On the other hand an irrational number when expressed as a decimal is neither terminating nor recurrent. These are deep statements which can be proved only with the tools of Higher Mathematics. It is therefore impossible to express irrational numbers — even such apparently simple ones as $\sqrt{2}$, $\sqrt{3}$ etc. — precisely in terms of terminating decimals or decimals with recurring parts. One can only approximate their actual value as closely as desired, by means of rational numbers.

Further it is not easy to perform the operations of addition and multiplication with irrational numbers. Of course we have been taught, for example, that

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}$$

and we shall certainly be using such relations all the time. But the proofs of these statements need precise definitions of irrational numbers. These precise definitions were given first by Dedekind in the 19th century. Just to give a broad picture of what his methods are, the definition of $\sqrt{2}$ goes as follows. Divide all the rational numbers (*i.e.*, the elements of **Q**) into two classes: the lower class L, and the upper class U. The lower class consists of all negative rationals and also those non-negative rationals whose squares are less than 2. The upper class consists of all those non-negative rationals whose squares are greater than 2. Both the classes are non-empty; because $1 \in L$ and $2 \in U$. Further their intersection is empty. Also every number in L is less than every number in U. So if we represent all the numbers on a geometric line, the above division of all rational numbers would be represented somewhat as follows:

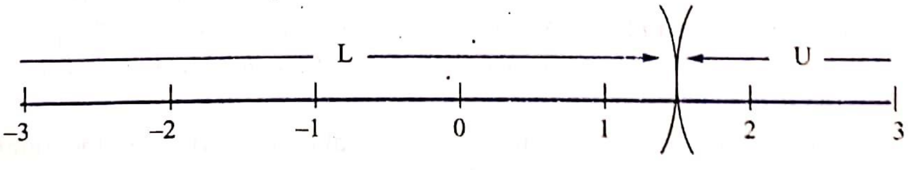


Fig. 1.1

Thus there is a gap between L and U. This gap was defined by Dedekind as the irrational number $\sqrt{2}$. The most proper way of establishing these results is to go with Dedekind and define irrational numbers by such 'cuts' of the geometric line. Dedekind not only defined them but also enunciated methods of addition and multiplication of

irrational numbers such that the class **R** consisting of all rational numbers and irrational numbers became a proper extension of **Q**. The *real number system* **R** is thus the geometric line, which is continuous and without gaps. This is the major advantage over the smaller system **Q**. The continuity of the real line (as it is called) means that every point on the line represents a real number and every real number has a positional representation on the line.

But again there is a 'price' for this extension. The elements of **Q** can be sequenced in the following sense. The whole set of rational numbers can be put into one-one correspondence with **N** that is, with

$$\{1, 2, 3, ..., n, ...\}.$$

This property is called 'countability'. But this fails in **R**. The fact that **Q** is countable whereas **R** is not is a major result which will form one of the interesting, foundational results in Advanced Mathematics. It is enough to say at this point that **R** has so many numbers in it that even n repetitions of **Q** (however large the number n may be) would still not come up in terms of size, to the number of elements of **R**. We say **R** is "uncountable".

Now we come to the *fourth* (and last) *stage* of this series of extensions. It is from **R** to **C** where **C** is the set of all *complex numbers* $x + iy$, $x, y \in$ **R**. Here i is the so-called 'imaginary' square root of –1. Note that one of the major deficiencies of **R** is that, within **R** we cannot solve several algebraic equations, the simplest of them being

$$x^2 + 1 = 0.$$

There is no $x \in$ **R** which is a solution of this equation, because $x^2 = -1$ is an impossible relation for any $x \in$ **R**; since the square of any real number is non-negative. So we invent a new number called i such that $i^2 = -1$. It is called 'imaginary' because it is not in the real number system and so it is not real!

There is nothing 'imaginary' about it in the English sense of the word. It has as much of an existence in the mind as any other number in mathematics. The number "2" for example is itself only a mental construct. There are two apples, two fingers, etc. in the concrete visual world, there are symbols for '2' which can be written and seen, but the number '2' by itself is only in the mind. **The number '*i*' also is as much of a mental construct and no more, as the number '2'.** The new number i is defined in such a way that it not only satifies $i^2 = -1$, but when adjoined to the system of real numbers it extends (**R**, +, .) into the bigger system (**C**, +, .) of complex numbers satisfying the associative, commutative, distributive properties already present in (**R**, +, .). Having defined i in this way we arrive at the general complex number $z = x + iy$ where x and y are real numbers. We note that (1) $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$. (2) $(a + ib) + (c + id) = (a + c) + i(b + d)$. (3) $(a + ib) \times (c + id) = (ac - bd) + i(ad + bc)$. (4) $\mathbf{R} \subseteq \mathbf{C}$ in the sense that if $x \in \mathbf{R}$ then $x = x + i0 \in \mathbf{C}$.

If $y = 0$, the number is x and therefore a (pure) real number. If $x = 0$, the number is iy and is called a wholly imaginary number. x is called the real part of the complex number z and is written as Re(z) or Re($x + iy$). The real number y is called the imaginary part of z and is written as Im(z) or Im($x + iy$). Thus we have, for every complex number $z = x + iy$

$$x = \text{Re}(z) = \text{Re}(x + iy)$$

and

$$y = \text{Im}(z) = \text{Im}(x + iy)$$

Also $x + iy$ and $x - iy$ are called conjugate complex numbers. Note that

$$(x + iy)(x - iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2.$$

By definition the sum of two complex numbers is a complex number and the product of two complex numbers is also a complex number. It is now a routine exercise to verify laws (1′), (2′), (3′), (4′) and (5′) for all $z \in$ **C** and also the special laws (6′) and (7′) for '0' and '1' in **C**. Note that the same 'zero' and the same '1' which worked for **Z**, **Q** and **R** also works here. Also at each stage of the extensions **N** to **Z**, **Z** to **Q**, **Q** to **R** whatever extra advantages we got, they are all present in **C**. In fact **C** not only extends (**R**, +, .) but extends them in a substantial manner. Any algebraic equation such as

$$a_0z^n + a_1z^{n-1} + ... + a_n = 0$$

with all a_i's in **C** has all its roots in **C**. This complete solvability of all algebraic equations is the major advantage of the extension from **R** to **C**. The result stating this complete solvability is known as the Fundamental Theorem of Algebra, whose proof requires quite a lot of higher mathematics.

But as before, we pay a price, and this time a big price for the extension. There is no order relation in **C** which extends the order relation in **R**. This is the loss we are prepared to put up with for the advantage gained in this last extension. But before we see the full force of this loss, we have to make an important observation.

The real line, as we know, has been designed in such a way that there are no gaps, in the sense we explained earlier. In this sense therefore, the real line is 'complete'. So when we introduce new numbers like the complex numbers for purposes of being able to solve more algebraic equations, we cannot expect to have geometrical representations of these numbers on the real line itself, keeping the earlier representation of numbers on the real line. So mathematics invented two 'copies' of the real line, one perpendicular to the other, the first one representing the real part of the complex number and the second one representing the imaginary part of the complex number. Such a representation of a complex number $z = x + iy$ as a point (x, y) on the coordinate plane is called an *Argand Diagram*. A point on the x-axis (now called the *Real axis*) is $(x, 0)$ and so represents the purely real numbers x; as a complex number it is nothing but $x + i0$. A point on the y-axis (now called the *Imaginary axis*) is $(0, y)$ and so represents the purely imaginary number iy. Note that, throughout in this discussion, x and y are real numbers.

If P is the point (x, y), it represents the complex number $z = (x + iy)$. We have the

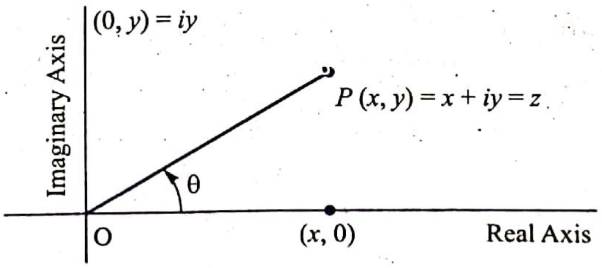


Fig. 1.2

following two fundamental definitions regarding $z = x + iy$ which is geometrically the same as P = (x, y).

Definition 3. The distance OP = $\sqrt{x^2 + y^2}$ from the origin is called the *modulus of z*. It is denoted by $|z|$ or $|x + iy|$. It is always non-negative.

For example $|2 + 3i| = 5$;

$|1 - i| = \sqrt{2}$ and so on.

Note that in particular the modulus of a real number x $(= x + i0)$ reduces to the following:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus $|-7| = -(-7) = 7 = |7|$. The modulus is also called the *absolute value*.

Definition 4. If θ is the angle which OP makes with the positive direction of the x-axis, θ is called the '*argument*' of z and is written arg z. We usually take $\theta = \arg z$ to lie between $-180°$ and $180°$, *i.e.*, such that $-180° < \theta° \leq 180°$.

Now since the complex numbers are geometrically all over the plane, there is no natural order among them, which will coincide with the natural order of the real numbers (= complex numbers $x + i0$) on the real line (= x-axis of the Argand Diagram), and which is properly compatible with the addition and multiplication in **C**. For instance, whatever way we design the order, either the number i has to be greater than the number zero or has to be less. Either way we get an incongruity with multiplication, since $i^2 = -1$ and this would mean that the L.H.S. here is a product of two quantities which are either both greater than 0 or both less than zero. In both the cases the R.H.S. being negative, we face a contradiction. Thus it is impossible to introduce an order in **C** which is compatible with multiplication in the above sense and which reduces to the natural order on the real line. Hence two complex numbers are either equal or unequal; there is no concept of greater or less.

This completes our outline of the five number systems **N**, **Z**, **Q**, **R** and **C**. We summarise in Table 1.1 the information about what is gained by each extension and what is lost. As we go from **N** to **Z** to **Q** to **R** to **C**, note that what is once gained remains a gain through all the further stages and what is once lost remains a loss in all the further stages.

EXAMPLE 1. *If $(a/b) < (c/d)$ with $b > 0$, $d > 0$ show that $(a + c)/(b + d)$ lies between a/b and c/d. (where a, b, c, d are real numbers).*

SOLUTION. If $(a/b) < (c/d)$ and b, d are positive then $ad < bc$ and hence

$$ab + ad < ab + bc.$$

This means that $a(b + d) < b(a + c)$ or $(a/b) < (a + c)/(b + d)$.

Table 1.1. Gain and Loss in the Extensions From N To Z To Q To R To C

<table>
<tr><th>Properties which are not present in N but gained in an extension</th><th>(1′)
$a + x = b$
is solvable</th><th>(2′)
$ax = b$,
$(a \neq 0)$ is
solvable</th><th>(3′)
No gaps in the geometrical construct which represents the numbers</th><th>(4′)
All algebraic equations are solvable</th></tr>
<tr><td></td><td>Z</td><td>Q</td><td>R</td><td>C</td></tr>
<tr><td>Properties which are present in N but lost in an extension</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">1. If $a > b$ then $xa > xb$ for all x.</td><td colspan="4">1′ is true</td></tr>
<tr><td colspan="4">1 is not true</td></tr>
<tr><td rowspan="2">2. There exists a next greater number for every number. (Induction)</td><td rowspan="2"></td><td colspan="3">2′ is true</td></tr>
<tr><td colspan="3">2 is not true</td></tr>
<tr><td rowspan="2">3. It is possible to arrange the entire system as an infinite sequence. (countability)</td><td rowspan="2"></td><td rowspan="2"></td><td colspan="2">3′ is true</td></tr>
<tr><td colspan="2">3 is not true</td></tr>
<tr><td rowspan="2">4. There exists a natural order relation compatible with addition and multiplication</td><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td><td>4′ is true</td></tr>
<tr><td>4 is not true</td></tr>
</table>

Similarly $ad < bc$ means $ad + cd < bc + cd$ or $(a + c)d < (b + d)c$. This means that $(a + c)/(b + d) < (c/d)$. Thus $(a/b) < (a + c)/(b + d) < (c/d)$.

EXAMPLE 2. *Let a and b be positive integers. Show that $\sqrt{2}$ always lies between (a b) and $(a + 2b)/(a + b)$.*

SOLUTION. Suppose $\sqrt{2} < (a/b)$. Then $2 < (a^2/b^2)$ or $2b^2 < a^2$. Therefore, we get

$$a^2 + 4b^2 < a^2 + a^2 + 2b^2 = 2a^2 + 2b^2$$

$$(a + 2b)^2 = a^2 + 4b^2 + 4ab < 2a^2 + 2b^2 + 4ab = 2(a + b)^2$$

$$\therefore \quad \{(a + 2b)/(a + b)\}^2 < 2 \text{ or } (a + 2b)/(a + b) < \sqrt{2}$$

On the other hand if $\sqrt{2} > (a/b)$ then $a^2 < 2b^2$.

$2(a + b)^2 = 2(a^2 + 2ab + b^2) = a^2 + a^2 + 2b^2 + 4ab < a^2 + 2b^2 + 2b^2 + 4ab = (a + 2b)^2$ or $\sqrt{2} < (a + 2b)/(a + b)$. Thus $\sqrt{2}$ always lies between a/b and $(a + 2b)/(a + b)$.

EXAMPLE 3. *Given any real number $x > 0$, show that there exists an irrational number ξ, such that $0 < \xi < x$.*

SOLUTION. If x is irrational, then choose $\xi = x/2$. Clearly $0 < \xi < x$.

If x is rational, then choose $\xi = x/\sqrt{2}$. Since $\sqrt{2} > 1$, we have $0 < \xi < x$.

(In fact there are infinitely many irrational numbers between any two real numbers.)

EXAMPLE 4. *Show that $\sqrt{2} + \sqrt{5}$ is irrational.*

SOLUTION. Suppose $\sqrt{2} + \sqrt{5} = x = p/q$ is a rational number with $p, q \in \mathbf{Z}$.

Then $(x - \sqrt{2})^2 = 5$ *i.e.* $x^2 - 2\sqrt{2}x + 2 = 5$.

Hence $x^2 - 3 = 2\sqrt{2}x$, which gives $\sqrt{2} = (x^2 - 3)/(2x)$, a rational number.

This contradicts the fact that $\sqrt{2}$ is rational. So $\sqrt{2} + \sqrt{5}$ is irrational.

EXAMPLE 5. *Express the rational number whose decimal expansion is 0.1234545454545 ... as ratio of two integers.*

SOLUTION. Let $x = 0.123454545 \ldots$

Then $10^3 x = 123.454545 \ldots$

$10^5 x = 12345.4545 \ldots$.

Subtracting, we get $(10^5 - 10^3)x = 12222.$

$x = 12222/99000.$

EXAMPLE 6. *Find all positive integers n for which $\sqrt{(n-1)} + \sqrt{(n+1)}$ is rational.*

SOLUTION. Let $x_n = \sqrt{(n-1)} + \sqrt{(n+1)}$ be rational. Then $1/x_n$ is also rational.

But $1/x_n = 1/(\sqrt{(n-1)} + \sqrt{(n+1)})$

$= (\sqrt{(n+1)} - \sqrt{(n-1)})/(\sqrt{(n-1)} + \sqrt{(n+1)})(\sqrt{(n+1)} - \sqrt{(n-1)})$

$= (\sqrt{(n+1)} + \sqrt{(n-1)})/2.$

This means $(\sqrt{(n+1)} - \sqrt{(n-1)}$ is also rational.

So $(\sqrt{(n-1)}$ and $\sqrt{(n+1)}$ are also rational.

i.e., $(n-1)$ and $(n+1)$ are perfect squares.

This is not possible as any two perfect squares differ at least by 3. Hence there is no positive integer n such that $(\sqrt{(n-1)} + \sqrt{(n+1)})$ is rational.

EXAMPLE 7. *If $a + \sqrt{b} = c + \sqrt{d}$, where a, b, c, d are rational then $a = c$ and $b = d$, unless b, d are squares of rationals.*

SOLUTION. Suppose $a \neq c$, let $a = c + x$. Then $a + \sqrt{b} = c + x + \sqrt{b} = c + \sqrt{d}$.

So $x + \sqrt{b} = \sqrt{d}$. Squaring we get, $d - x^2 - b = 2x\sqrt{b}$. This implies that $\sqrt{b}$ is rational, hence $\sqrt{d}$ is also rational. Thus b and d are squares of rationals. Hence the result.

EXAMPLE 8. *If $a + b(\sqrt[3]{p}) + c(\sqrt[3]{p^2})$, where a, b, c, p are rational and p is not a perfect cube, then a, b, c are all zero.*

SOLUTION. We have $$a + b(\sqrt[3]{p}) + c(\sqrt[3]{p^2}) = 0 \tag{1}$$

Therefore, $$a(\sqrt[3]{p}) + b(\sqrt[3]{p^2}) + cp = 0 \tag{2}$$

Now $b \times (1) - c \times (2)$ gives

$$(b^2 - ac)\sqrt[3]{p} + ab - c^2p = 0 \tag{3}$$

$\sqrt[3]{p}$ is irrational and therefore from (3) we get $b^2 - ac = 0$ and $ab = c^2p$.

$\therefore$ $c^4p^2 = a^2b^2 = a^3c$. If $c \neq 0$, then we get $p^2 = a^3/c^3$ which is not true as $\sqrt[3]{p}$ is irrational. $\therefore$ $c = 0$ which in turn implies that $a = b = 0$.

EXAMPLE 9. *If a, b, c, d are all rational and, $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ then show that either (i) $a = c$ and $b = d$ or (ii) $a = d$ and $b = c$ or (iii) the quotients $\sqrt{(a/b)}$, $\sqrt{(a/c)}$, $\sqrt{(a/d)}$, $\sqrt{(b/c)}$, $\sqrt{(b/d)}$, $\sqrt{(c/d)}$, are all rational.*

SOLUTION. $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ gives on squaring $a + b = c + d$ and $\sqrt{ab} = \sqrt{cd}$ unless $\sqrt{ab}$, $\sqrt{cd}$ are rational. (See Example 7.)

$\therefore$ $(\sqrt{a} - \sqrt{b})^2 = (\sqrt{c} - \sqrt{d})^2$ unless $\sqrt{ab}$, $\sqrt{cd}$ are rational. This means that $|\sqrt{a} - \sqrt{b}| = |\sqrt{c} - \sqrt{d}|$ unless $\sqrt{ab}$ and $\sqrt{cd}$ are rational.

Case 1. $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ and $\sqrt{a} - \sqrt{b} = \sqrt{c} - \sqrt{d}$ gives $a = c$ and $b = d$.

Case 2. $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ and $\sqrt{a} - \sqrt{b} = \sqrt{d} - \sqrt{c}$ gives $a = d$ and $b = c$.

That $\sqrt{ab}$ and $\sqrt{cd}$ are rational implies that $\sqrt{(a/b)}$ and $\sqrt{(c/d)}$ are rational.

$\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ also implies $\sqrt{a} - \sqrt{c} = \sqrt{d} - \sqrt{b}$, which gives $a = d$ and $b = c$ or $\sqrt{ac}$, $\sqrt{bd}$ are rational. Finally we conclude that either

(*i*) $a = c$ and $b = c$ or (*ii*) $a = d$ and $b = c$ or (*iii*) the quotients $\sqrt{(a/b)}$, $\sqrt{(a/c)}$, $\sqrt{(a/d)}$, $\sqrt{(b/c)}$, $\sqrt{(b/d)}$, $\sqrt{(c/d)}$ are all rational.

EXAMPLE 10. *Find a polynomial equation of the lowest degree with rational coefficients of which one root is* $\sqrt[3]{2} + 3\sqrt[3]{4}$.

SOLUTION. Let $\quad x = \sqrt[3]{2} + 3\sqrt[3]{4}$.

Then we have $\quad x^3 = 2 + 108 + 18(\sqrt[3]{2} + 3\sqrt[3]{4})$

$\quad = 110 + 18x.$

$\therefore \quad x^3 - 18x - 110 = 0$

(It is clear from Example 8 that no quadratic expression in x with rational coefficients becomes 0. So the least degree is 3.)

CHAPTER

2

ARITHMETIC OF INTEGERS

2.1 THE PRINCIPLE OF INDUCTION

In this chapter we shall see certain fundamental properties valid in the number systems **N** and **Z**.

Consider a statement about the positive integers.

For example (1) $n(n+1)(n+2)$ is always divisible by 6

(2) The sum of the first n natural numbers is given by

$$S_n = \frac{n(n+1)}{2}$$

(3) $2^n > n$ for all natural numbers

(4) $(1+x)^n$

$$= 1 + nx + \frac{n(n+1)}{1.2}x^2 + \ldots$$

$$+ \frac{n(n-1)\,(n-1)\,(n-2)\ldots(n-r_{n+1})}{1.2.3\ldots.r}x^r + \ldots + x^n$$

for all positive integers

are all statements about positive integers n. If one wants to check the validity of these statements, how should one go about it?

Take the problem of finding the sum S_n of the first n natural numbers. The statement (2) above says that $S_n = \frac{n(n+1)}{2}$ for all n. When we try verifying with the first few natural numbers, we observe that $S_1 = 1$, $S_2 = 3$, $S_3 = 6$, $S_4 = 10$, $S_5 = 15$ all satisfy $S_n = \frac{n(n+1)}{2}$. But from a few verifications, or for that matter, from any number of verifications, one cannot conclude that the result is always true. So, what do we do? We observe that

$$S_{k+1} = 1 + 2 + 3 + \ldots + k + (k+1) = S_k + (k+1).$$

If we have checked and found, or if we assume that the formula

$S_n = \frac{n(n+1)}{2}$ is true for all $n \leq k$, the equation $S_{k+1} = S_k + (k+1)$ gives

$$S_{k+1} = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

This says that if the formula is true for $n = k$, then it has to be true for the next integer $k+1$. We have checked that the result is true for $k = 1$. Therefore, by what we have just seen, it must be true for $k = 2$ and hence for the next integer 3. Proceeding inductively, we see that the formula has to be true for all positive integers. Thus we have essentially proved a very important formula, namely, the sum of the first n natural numbers is given by $S_n = \dfrac{n(n+1)}{2}$.

The underlying mathematical principle in the above argument is called the **'Principle of Mathematical Induction'**. It can be stated as follows.

Let $P(n)$ be a statement about the positive integers such that

(1) $P(1)$ is true, *i.e.*, the statement is true for $n = 1$.

(2) Whenever the statement is true for $n = k$, it is true for $n = k + 1$. Then $P(n)$ is true for all natural numbers n.

This principle is one of the fundamental principles in mathematics and it is a tool indispensable in most of the branches of mathematics.

REMARK The above principle works because of the fact that every nonempty subset of positive integers has a least element. In fact, it can be proved that the principle of mathematical induction is equivalent to the fact that every nonempty set of positive integers has a smallest element.

EXAMPLE 1. *If x is any real (complex) number such that $x \neq 1$ then $1 + x + x^2 + \ldots + x^{n-1} = S_n = \dfrac{1-x^n}{1-x}$ for every positive integer n.*

SOLUTION. We prove this by induction on n. As a first step, we check that $S_1 = 1 = (1-x)/(1-x)$. Therefore the result is true for $n = 1$. Suppose we assume that it is true for $n = k$. Then $1 + x + x^2 + \ldots + x^{k-1} = S_k = \dfrac{1-x^k}{1-x}$.

$$\therefore \quad S_{k+1} = 1 + x + x^2 + \ldots + x^k = S_k + x^k$$

$$= \frac{(1-x^k)}{(1-x)} + x^k = \frac{1-x^{k+1}}{1-x}.$$

Thus, whenever the formula is true for k, it is true for $k + 1$; and it is found to be true for $k = 1$. Hence by the principle of induction $S_n = \dfrac{1-x^n}{1-x}$ for all n.

Note. This example is actually that of what is called a *geometric progression*. See Chapter 15 for more on this.

EXAMPLE 2. *$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$, where n is any positive integer greater than one.*

SOLUTION. The formula for $n = 2$ reads $x^2 - y^2 = (x-y)(x+y)$ which is true. Assume now that the result is true for $n = k$ for some $k > 2$. This gives

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \ldots + xy^{k-2} + y^{k-1}).$$

Therefore $x^{k+1} - y^{k+1}$

$$= x(x^k - y^k) + y^k(x - y)$$
$$= x(x - y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \ldots + xy^{k-2} + y^{k-1}) + y^k(x - y)$$

(by our induction hypothesis)

$$= (x - y)\ \{x(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \ldots + xy^{k-2} + y^{k-1}) + y^k\}$$
$$= (x - y)\ (x^k + x^{k-1}y + x^{k-2}y^2 + \ldots + xy^{k-1} + y^k).$$

Thus the formula is true for $k + 1$ whenever it is true for any k. We have checked that it is true for $k = 2$. Therefore by the principle of induction the formula is true for all positive integers $n \geq 2$.

EXAMPLE 3. *Prove that*

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

SOLUTION. When $n = 1$, the left hand side reads $1/1.2 = 1/2$ which is the same as $n/n + 1$ when $n = 1$. Thus the result is true for $n = 1$. Assume now that the result is true for $n = k$. This gives

$$S_k = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Therefore
$$S_{k+1} = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= S_k + \frac{1}{(k+1)\,(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)\,(k+2)} \qquad \text{(by the induction hypothesis)}$$
$$= \frac{k(k+2)+1}{(k+1)\,(k+2)} = \frac{k^2+2k+1}{(k+1)\,(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)\,(k+2)} = \frac{k+1}{k+2}.$$

(Note that the cancellation of $(k + 1)$ is valid here as $k + 1 \neq 0$). We have now proved that whenever the result is true for $n = k$ it is also true for $n = k + 1$; and we have already checked it to be true for $n = 1$. Hence, by the principle of induction the result is true for all positive integers.

EXAMPLE 4. *If there are n participants in a knock-out tournament then prove that (n – 1) matches will be needed to declare the champion.*

SOLUTION. When $n = 1$, there is no match needed and the result is trivially true. This gives a base for induction. Suppose $n = 2k$ for some positive integer k. Then we have k matches played in the first round of the tournament. The k winners should play $(k - 1)$ matches to find the champion, if we make the induction hypothesis that the result is true for all integers $m < n$. Therefore in all we require $k + k - 1 = 2k - 1 = n - 1$ matches to be played. Suppose $n = 2k + 1$ with $k \geq 1$. Again in the first round of k matches we have k losers. The remaining $(k + 1)$ men have to play k matches to find the champion, again using the induction hypothesis made above. Thus we require in all

$$k + k = 2k = n - 1$$

matches to be played.

EXAMPLE 5. *Prove that $5^{2n} - 6n + 8$ is divisible by 9 for all positive integers n.*

SOLUTION. If $f(n) = 5^{2n} - 6n + 8$ then $f(1) = 5^2 - 6 + 8 = 27$ which is divisible by 9. Therefore the result is true for $n = 1$. Assume that $f(n)$ is divisible by 9 for some $n > 1$. Then we have

$$\begin{aligned} f(n+1) &= 5^{2(n+1)} - 6(n+1) + 8 = 5^{2n} \,.\, 5^2 - 6(n+1) + 8 \\ &= 5^2(5^{2n} - 6n + 8) + 144n - 198 \\ &= (\text{Multiple of } 9) + 9(16n - 22), \quad \text{(by induction hypothesis)} \\ &= \text{Multiple of } 9 \end{aligned}$$

Therefore by the principle of induction 9 divides $f(n)$ for all positive integers n.

EXAMPLE 6. *Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$. Show that*

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \qquad \textit{for integers } n \geq 1.$$

SOLUTION. For $n = 1, 2$ we have

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = 1 \qquad \text{(readily checked).}$$

Assume the result to be true for all integers k such that $1 \leq k \leq n$.

For $n \geq 2$ we have

$$\begin{aligned} u_{n+1} &= u_n + u_{n-1} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\ &\qquad\qquad \text{(by induction hypothesis)} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \end{aligned}$$

Thus, whenever the result is true for $k \leq n$, we see that it is true for $k = n + 1$. Therefore, by the principle of mathematical induction, the result is true for all positive integers.

EXERCISE 2.1

1. Prove that $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ using the induction principle.
2. Using induction prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$ for each positive integer n.
3. Prove that $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.
4. Let $a_1 = a_2 = 1$, $a_3 = 2$ and $= a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. The sequence (a_n) is known as the Fibonacci sequence. Prove that
   (i) $a_1 + a_2 + ... + a_n = a_{n+2} - 1$.
   (ii) $a_1 + a_3 + a_5 + ... + a_{2n-1} = a_{2n}$.
   (iii) $a_2 + a_4 + a_6 + ... + a_{2n} = a_{2n+1} - 1$.
   (iv) $a_{n+1}^2 = a_n a_{n+2} = (-1)^n$.
   (v) $a_1a_2 + a_2a_3 + ... + a_{2n-1}\, a_{2n} = (a_{2n})^2$.
   (vi) $a_1a_2 + a_2a_3 + ... + a_{2n}\, a_{2n+1} = (a_{2n+1})^2 - 1$.
5. Define (b_n) by $b_1 = 1$, $b_n = a_{n+1} - a_n$ for $n \geq 2$. (b_n) is known as the sequence of Lucas numbers.

   Prove
   (i) $b_n = b_{n-1} + b_{n-2}$ for $n \geq 3$.
   (ii) $a_{2n} = a_n b_n$.
   (iii) $b_1 + 2b_2 + 4b_3 + 8b_4 + ... + 2^{n-1}\, b_n = 2^n a_{n+1} - 1$.

   where (a_n) is the Fibonacci sequence of numbers defined in exercise 4.
6. Prove by induction that the product $n(n + 1)(n + 2) \ldots (n + r - 1)$ of any consecutive r numbers is divisible by $r!$.
7. If $S_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ show that S_n is an integer and that $S_{n+1} = 6S_n - 4S_{n-1}$. Deduce that the next integer greater than $(3 + \sqrt{5})^n$ is divisible by 2^n.
8. Show that n^2 is the sum of the first n consecutive odd numbers.
9. Prove by induction that $2^n > n^3$ if $n > 9$.
10. Show that $2.7^n + 3.5^n - 5$ is divisible by 24 for all positive integers n.
11. If m, n, p, q are non negative integers prove that

$$\sum_{m=0}^{q} (n-m)\frac{(p+m)!}{m!} = \frac{(p+q+1)!}{q!}\left(\frac{n}{p+1} - \frac{q}{p+2}\right).$$

12. Prove $\dfrac{k^7}{7} + \dfrac{k^5}{5} + \dfrac{2k^3}{3} - \dfrac{k}{105}$ is an interger for every positive integer k.

2.2 DIVISIBILITY

The equation $ax = b$, $a \neq 0$ does not always have a solution in **Z** the set of integers. For example $3x = 12$ has a solution $x = 4$ but $5x = 12$ does not have any solution in **Z**. This observation prompts the following definition.

Definition 1. The integer a *divides* an integer b if there exists an integer c such that $ac = b$.

In other words a divides b in **Z** if $ax = b$ has a solution for x in **Z**. When a divides b we call b a multiple of a. For example, the multiples of 2 are $0, \pm 2, \pm 4, \pm 6, \pm 8$ ... which we call the set of even numbers. A number is even if it is a multiple of 2. Given any

integer a, we always have $a.0 = 0$ and therefore 0 is a multiple of every integer, or every integer divides 0. In fact, 0 is the only such integer. When a divides b we also say that a is a divisor of b. For example, the divisors of 2 are just $\pm 1, \pm 2$; the divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12. At times we also say 'factors' in the place of 'divisors'. We observe that ± 1 are divisors of every other integer. Given any integer a, it has at least two factors 1 and a.

Notation. $a \mid b$ means that a divides b.

Proposition 1. If a divides b and b divides c then a divides c.

Proof. a divides b implies that there exists an integer k such that $ak = b$. Also b divides c implies that there exists an integer l such that $bl = c$. This gives $c = bl = (ak)l. = a.kl$ which implies that a divides c as kl is an integer whenever k and l are integers. ❑

Proposition 2. For any integer k let $k\mathbf{Z} = \{0, \pm k, \pm 2k, \pm 3k \ldots\}$ denote the set of multiples of k. Then a divides b implies that $a\mathbf{Z} \supseteq b\mathbf{Z}$, *i.e.*, every multiple of b is also a, multiple of a.

Proof. If $c \in b\mathbf{Z}$ then b divides c; now we have, a divides b and b divides c. Therefore by Proposition 1, a divides c, which implies that $c \in a\mathbf{Z}$. Thus $b\mathbf{Z} \subseteq a\mathbf{Z}$ whenever a divides b. ❑

For example, 3 divides 6 and 6 divides 12. We have and $3\mathbf{Z} \supseteq 6\mathbf{Z} \supseteq 12\mathbf{Z}$. (*i.e.*, the set of multiples of 3 contains the set of multiples of 6 and which in turn contains the set of multiples of 12).

Proposition 3. $a\mathbf{Z} = \mathbf{Z}$ if and only if $a = \pm 1$.

Proof. The above statement means the following. $a\mathbf{Z} = \mathbf{Z}$ implies that $a = \pm 1$ and conversely $a = \pm 1$ implies that $a\mathbf{Z} = \mathbf{Z}$. In fact, this statement follows immediately from the fact that ± 1 are the only integers which are divisors of every other integer. If $a \neq \pm 1$ we have $a\mathbf{Z}$ as a proper subset of $\mathbf{Z}$. ❑

Now 7 divides 21 and 7 divides 35 and we have 7 dividing $21x + 35y$ for any two integers x and y. In fact, in general we have the following proposition.

Proposition 4. If a divides b and a divides c in $\mathbf{Z}$ then a divides $xb + yc$ for any integers x, y in $\mathbf{Z}$ (In other words a divides every integral linear combination of b and c).

Proof. a divides b and a divides c imply that there exist integers k and l such that $ak = b$ and $al = c$. Therefore $xb + yc = x(ak) + y(al) = a(xk + yl)$. This means that a divides $xb + yc$. ❑

We note that certain integers have a large number of factors, while some others have only a few factors. We have already noted that every integer $x > 0$ has at least two positive factors, namely 1 and x. Some positive integers x have just these two as their positive factors. These integers play an important role in Mathematics.

Definition 2. A positive integer p is a *prime* if $p \neq 1$ and the only positive divisors of p are 1 and p.

For example 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 ..... are the first few primes.

If a and b are two integers then any integer c that divides both a and b is called a common divisor of a and b.

For example (*i*) the common divisors of 4 and 8 are $\pm 1, \pm 2, \pm 4$.

(*ii*) the common divisors of 8 and 12 are $\pm 1, \pm 2, \pm 4$.

(*iii*) the common divisors of 12 and 35 are ± 1.

(*iv*) the common divisors of 7 and 24 are ± 1.

(*v*) the common divisors of 24 and 60 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Definition 3. If a and b are integers such that not both of them are zero, then a positive integer d is called *the greatest common divisor* (written g.c.d.) of a and b if

(*i*) d is a common divisor of a and b

(*ii*) each integer c that divides both a and b also divides d.

The first question that comes to our mind is that given a pair of integers, not both zero, should there exist a greatest common divisor, and when it exists, should it be unique, justifying the definite article 'the' used in the above definition. We try to answer these questions, by studying some examples.

EXAMPLE 1. (*i*) *The positive common divisors of 4 and 8 are 1, 2 and 4. Therefore the g.c.d. of 4 and 8 is 4.*

(*ii*) *The only positive common divisor of 12 and 35 is 1 and therefore the g.c.d. of 12 and 35 is 1.*

(*iii*) *The positive common divisors of 24 and 60 are 1, 2, 3, 4, 6 and 12. Therefore the g.c.d. of 24 and 60 is 12.*

SOLUTION. In these examples, we could actually enumerate the common divisors. If the numbers are big, such an enumeration becomes extremely difficult. Given two integers a and b we look for smaller integers a_1 and b_1 which have the same g.c.d. Consider 138 and 1239. We have $1239 = (8 \times 138) + 135$. Now any common divisor of 1239 and 138 has to divide 135 as seen from the equation $135 = 1239 - 8\,(138)$. Also from $1239 = 8(138) + 135$ we see that any common divisor of 138 and 135 also divides 1239. Thus the set **S** of common divisors of 1239 and 138 is the same as the set of common divisors of 138 and 135. Now 138 and 135 are smaller compared to 1239 and 138. Again writing $138 = 1(135) + 3$ we note that S = the set of common divisors of 135 and 3, given by $S = \{\pm 1, \pm 3\}$.

Thus the g.c.d. of 1239 and 138 is 3. Such a simplification was possible because we could divide 1239 by 138 to get a quotient and a remainder smaller than 138. That such a division is possible for any two integers $a > 0$ and b is the following division algorithm due to Euclid.

Theorem 1. (*Euclid's division lemma* or *the division algorithm*)

For any integers $a > 0$ and b there exist unique integers q and r such that $b = aq + r$ with $0 \le r < a$.

Proof. Consider the representation of the integers on a geometric line. Plot the points

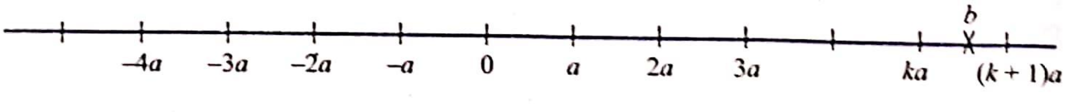


Fig. 2.1

corresponding to the multiples of a. If b is a multiple of a then it is of the form $b = ka$ for some integer k and we may take $q = k$ and $r = 0$. Otherwise, we can find two multiples ka and $(k + 1)a$ of a such that $ka < b < (k + 1)a$. Hence $0 \le b - ka < (k + 1)a - ka = a$. Take $q = k$, $r = b - ka$ to get $b = aq + r$ with q, r being integers and $0 \le r < a$.

Now we prove the uniqueness of q and r. Suppose $b = aq_1 + r_1 = aq_2 + r_2$ with q_1, q_2, r_1, r_2 being integers and $0 \le r_1, r_2 < a$. Then we should have $0 = (aq_1 + r_1) - (aq_2 + r_2)$ which implies that $a(q_1 - q_2) = r_2 - r_1$. We may assume $r_1 \le r_2$. This gives $a(q_1 - q_2) = r_2 - r_1 \le r_2 < a$ which means that a divides $r_2 - r_1$ where $r_2 - r_1$ is a non-negative integer less than a. This is possible only if $r_2 - r_1 = 0$ which in turn implies that $aq_1 = aq_2$. Now $aq_1 = aq_2$ gives $0 = a(q_1 - q_2)$ and this implies that $q_1 - q_2 = 0$ since by our assumption $a > 0$. This proves that $r_1 = r_2$ and $q_1 = q_2$, and the uniqueness is established.

❑

Note. Here in this proof we have used the fact that given $a > 0$ in **Z** and any integer b we can always find $k \in$ **Z** such that $ka > b$, which is more or less evident from the geometric representation on a straight line. For a purely algebraic proof see the exercises.

Notation. We write (a, b) for the g.c.d of a and b. We note that if $a \ne 0$ and a divides b, then $(a, b) = |a|$.

EXAMPLE 2. *Find the g.c.d. of 341 and 403.*

SOLUTION. Dividing 403 by 341 we get 403 = 1(341) + 62. Now the common divisors of 341 and 403 are precisely the common divisors of 341 and 62 (as seen already in the discussion just preceding Euclid's algorithm). Again dividing 341 by 62 we get 341 = 5(62) + 31 and hence the common divisors of 341 and 62 are precisely the common divisors of 62 and 31. Now the common divisors of 31 and 62 are $\pm 1, \pm 31$.

Thus we have
$$403 = 1\,(341) + 62$$
$$341 = 5\,(62) + 31$$
$$62 = 2(31) + 0$$

Hence $$(341, 403) = (341, 62) = (62, 31) = 31.$$

The above example tells us that there is nothing special about 341 and 403; we could have calculated the g.c.d. of any two integers a, b not both zero simultaneously, by the same algorithmic process. This observation leads to the following theorem.

Theorem 2. If a and b are integers, not both simultaneously zero, then (a, b) exists and is unique.

Proof. From the definition, it is clear that the g.c.d. of a and b is not affected by their signs, *i.e.*, $(a, b) = (a, -b) = (-a, b) = (-a, -b)$ whenever they exist. If one of them, say $b = 0$, then $(a, b) = |a|$ since every integer $a \ne 0$ divides 0. Therefore we assume that $a \ge b > 0$. We apply Euclid's division algorithm repeatedly to get

$$\begin{array}{ll} a = bq_1 + r_1 & 0 \le r_1 < b \\ b = r_1q_2 + r_2 & 0 \le r_2 < r_1 \\ r_1 = r_2q_3 + r_3 & 0 \le r_3 < r_2 \\ \vdots & \vdots \\ r_{n-3} = r_{n-2}q_{n-1} + r_{n-1} & 0 \le r_{n-1} < r_{n-2} \\ r_{n-2} = r_{n-1}q_n + r_n & 0 \le r_n < r_{n-1} \end{array} \qquad (1)$$

Now $b > r_1 > r_2 > r_3 > \ldots \ge 0$ says that this process has to end up in at most b steps; and we will obtain the corresponding $r_n = 0$. Therefore the last equation in the above sequence should read

$$r_{n-2} = r_{n-1}q_n + 0. \qquad (2)$$

By construction, each $r_i > 0$ for $i = 1, 2, \ldots, (n-1)$ and therefore $r_{n-1} > 0$. Retracing back from the last equation (2) in the system of equations (1) we see that r_{n-1} divides

$r_{n-2}, r_{n-3}, \ldots b$ and finally a. Therefore r_{n-1} is a common factor of a and b. Suppose d is any other common factor of a and b. Then, again the system of equations (1) tells us that d divides $r_1, r_2, r_3, \ldots, r_{n-2}$ and r_{n-1}. Thus r_{n-1} is a g.c.d. of a and b, proving the existence part of the theorem. To prove the uniqueness part, assume that d_1 and d_2 are both greatest common divisors of a and b. Then d_1 is a g.c.d. and d_2 is a common divisor implies that d_2 divides d_1. Similarly, reversing the roles of d_1 and d_2 we see that d_1 divides d_2. This means that there exist integers k, l such that $d_1 = kd_2$ and $d_2 = ld_1$. This says that

$$d_1 = kd_2 = k.ld_1 = kl.d_1.$$

This implies that $kl = l$ which is possible only if $k = l = 1$. We note that $k = l = -1$ is not possible as d_1 and d_2 are positive integers by our definition of a greatest common divisor. Thus $d_1 = d_2$, proving the uniqueness of g.c.d. of two numbers. ❑

EXAMPLE 3. *We again go back to Example 2 of finding the g.c.d. of 341 and 403.*

SOLUTION. We had the following equations.

$$403 = 1(341) + 62 \qquad (3)$$

$$341 = 5(62) + 31 \qquad (4)$$

$$62 = 2(31) \qquad (5)$$

We start from the last but one equation, namely (4) and rewrite this as

$$31 = 341 - 5(62) \qquad (6)$$

$$= 341 - 5(403 - 341) \text{ from (3)}$$

$$= 6(341) - 5(403) \qquad (7)$$

This expresses (403, 341) as an integral linear combination of 403 and 341; namely

$$31 = 6(341) + (-5)(403).$$

We also have $\quad 31 = 6(403) - 7(341)$

$$= 17(403) - 20(341).$$

In general, there may be many such pairs of integers x and y such that $(a, b) = xa + yb$. This example suggests the following theorem.

Theorem 3. If $d = (a, b)$, then there exist integers x and y such that $d = xa + yb$.

Proof. The theorem is a corollary of the theorem on Euclid's algorithm. Essentially, we have to do all that we have done in Example 3 for the general case.

$$a = bq_1 + r_1$$

$$b = r_1q_2 + r_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$r_{n-3} = r_{n-2}\,q_{n-1} + r_{n-1}$$

$$r_{n-2} = r_{n-1}\,q_n \qquad (8)$$

It is not hard to imitate what we have done in the example to get (a, b)

$$= r_{n-1} = xa + yb \text{ for some suitable integers } x \text{ and } y.$$

Note. We saw in Example 3, that (a, b) may be written as an integral linear combination of a and b in more than one way.

Corollary An integer c is a linear combination of two integers a and b in the form $c = xa + yb$ with x, y in **Z** if and only if $d = (a, b)$ divides c.

Proof. Let $a = kd$ and $b = ld$ with k, l in **Z**. Then $c = xa + yb$ implies that $c = xkd + yld$ or d divides c. Conversely, suppose d divides c. Then there exists an integer n such that

$c = nd$. By our theorem, there exist integers x' and y' such that $(a, b) = d = x'a + y'b$. This gives $c = nd = (nx')\,a + (ny')\,b = xa + yb$ where $x = nx' \in \mathbf{Z}$ and $y = ny' \in \mathbf{Z}$.

EXAMPLE 4. Whenever 3 divides an integer n and 4 also divides n, we see that $3 \times 4 = 12$ divides n. Does it mean that 'a divides n and b divides n' always imply that ab divides n ? Now 6 divides 36 and 4 divides 36; but $6 \times 4 = 24$ does not divide 36. Therefore the answer to the above question is that it is not always true that ab divides n. If we analyse the pairs 3, 4 and 6, 4 we see that every common multiple of 3, 4 is a multiple of their product; but not so for the other pair 6, 4. This is because $(3, 4) = 1$ and $(6, 4) = 2 > 1$. This motivates the following definition.

Definition 4. Two integers a and b are *relatively prime* if $(a, b) = 1$. In this situation we also say they are *coprime*.

We note that a and b are relatively prime if and only if their only common factors are ± 1.

For example (*i*) 12 and 35 are relatively prime

(*ii*) 6 and 17 are relatively prime

(*iii*) 12 and 18 are not relatively prime as $(12,18) = 6$

(*iv*) a prime number p is relatively prime to every integer n which is not a multiple of p. This is because the only positive divisors of a prime p are 1 and p. In particular two primes are always relatively prime.

(*v*) If $d = (a, b)$ then $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime. One can easily see this by writing $d = xa + yb$ for some integers x, y. Therefore $x\frac{a}{d} + y\frac{b}{d} = 1$. So, by the Corollary to Theorem 3, $\left(\frac{a}{d}, \frac{b}{d}\right)$ divides 1.

This means that $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

The following theorem justifies the observations made in the discussion of Example 4.

Theorem 4. If an integer c divides the product ab of two integers a and b and if a and c are relatively prime then c divides b. In other words if $c \mid ab$ and $(c, a) = 1$ then $c \mid b$.

Proof. We may write $(c, a) = 1 = xa + yc$ for some integers x and y.

c divides ab implies that $ab = ck$ for some $k \in \mathbf{Z}$. Therefore $b = b.\,1 = b\,(xa + yc) = xab + byc = xkc + byc$. This gives $b = c\,(kx + by)$ or c divides b. ❑

Caution In general if c divides ab it is not necessary that c divides a or c divides b. For example 6 divides $24 = 8 \times 3$; but 6 divides neither 8 nor 3. However, we have

Theorem 5. If a and b are integers, p is a prime that divides ab and p does not divide a, then p has to divide b.

Proof. Since p is a prime, if p does not divide a then $(a, p) = 1$. In other words p and a are relatively prime. Therefore by Theorem 4, p divides b. ❑

Corollary If p is a prime and p divides a product of integers, then p divides at least one of them.

Proof. Let p be a prime dividing the product $a_1, a_2 \ldots.\, a_n$ of integers. Now $p \mid a_1\,(a_2\, a_3 \ldots a_n)$ implies, by the theorem, that p divides a_1 or p divides the product $a_2\,a_3 \ldots a_n$. If p does not divide a_1 we have p dividing $a_2(a_3\,a_4 \ldots a_n)$. This gives p divides a_2 or a_3

$a_4 \dots a_n$. Thus by a repeated application of the theorem we see p divides a_i for some $i \in \{1, 2, 3, \dots n\}$. ❑

EXAMPLE 5. *For any positive integer m we have*

$$(ma, mb) = m\,(a, b).$$

SOLUTION. We note that (a, b) = least positive value of $\{ax + by \mid x, y \in \mathbf{Z}\}$. This follows from the Corollary to Theorem 3.

$\therefore \quad (ma, mb)$ = least positive value of $\{max + mby \mid x, y \in \mathbf{Z}\}$

$= m$. least positive value of $\{ax + by \mid x, y \text{ in } \mathbf{Z}\}$

$= m\,(a, b)$.

EXAMPLE 6. *If d divides a, d divides b and if d > 0 then*

$$\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}\,(a, b)$$

SOLUTION. Now $d > 0$, and hence by Example 5,

$$d \cdot \left(\frac{a}{d}, \frac{b}{d}\right) = (a, b)$$

or,

$$\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}\,(a, b).$$

EXAMPLE 7. *If (a, n) = (b, n) = 1 then (ab, n) = 1 i.e., If a and b are relatively prime to n then so is ab.*

SOLUTION. We can find integers x_1, y_1, x_2, y_2 such that

$$ax_1 + ny_1 = 1 = bx_2 + ny_2, \text{ since } (a, n) = 1 = (b, n).$$

$$\therefore \quad (ax_1)(bx_2) = (1 - ny_1)(1 - ny_2) = 1 - n(y_1 + y_2 - ny_1y_2)$$

$$\therefore \quad ab\, x_1 x_2 + n(y_1 + y_2 - ny_1y_2) = 1 \text{ and hence } (ab, n) = 1 \text{ (why?)}$$

EXAMPLE 8. *For any integer x we have*

$$(a, b) = (b, a) = (a, -b) = (a, b + ax).$$

SOLUTION. We have $(a, b) = (b, a) = (a, -b) = (-a, -b)$.

Suppose $(a,b) = d_1$ and $(a, b + ax) = d_2$. Then d_1 divides a, d_1 divides b and hence d_1 divides $b + ax$. This means that d_1 divides d_2. Similarly we see that d_2 divides d_1. Therefore $d_1 = d_2$.

Definition 5. If $a_1, a_2, \dots, a_n$ are all different from zero, the least of all the positive common multiples of $a_1, a_2, \dots, a_n$ is *the least common multiple* or the *l.c.m.* of $a_1, a_2, \dots, a_n$. We write the *l.c.m.* as $[a_1, a_2, \dots, a_n]$.

Thus $[a_1, a_2, \dots, a_n] = \min\{x > 0 \mid a_i \text{ divides } x \text{ for } i = 1, 2, \dots, n\}$

For example $[6, 9] = 18$, $[5, 7] = 35$, $[12, 18] = 36$.

EXAMPLE 9. *If x is any common multiple of $a_1, a_2, \dots, a_n$ all different from zero then $[a_1, a_2, \dots, a_n]$ divides x.*

SOLUTION. Let $[a_1, a_2, \dots, a_n] = a$. We write $x = aq + r$ with q, r being integers and $0 \le r < a$. Now a_i divides x and a_i divides a for each i. This means that a_i divides $x - aq = r$ for each i. But $0 \le r < a$ = *l.c.m.* of $(a_1, a_2 \dots a_n)$ implies that $r = 0$. Therefore $x = aq$ or a divides x.

This example shows that if $a = [a_1, a_2, \dots, a_n]$ then

$\{0, \pm a, \pm 2a \dots, \pm na, \dots\}$ is the set of all common multiples of $a_1, a_2, \dots, a_n$.

EXAMPLE 10. *For any $m > 0$, $[ma, mb] = m. [a, b]$*

SOLUTION. $[ma, mb]$ is clearly a multiple of m. Let $[ma, mb] = km$ and $[a, b] = l$. Now a divides l, and b divides l implies that ml is a common multiple of ma and mb. Therefore $[ma, mb] = mk$ divides ml or *equivalently* k divides L.

Also mk is a common multiple of ma, mb implies that k is a common multiple of a, b. Therefore l divides k. Thus we have k divides l and l divides k. Further both k and l are positive. Hence $k = l$. In other words $[ma, mb] = mk = ml = m[a, b]$.

EXAMPLE 11. *For any two non zero integers a and b we have $[a, b]\,(a, b) = |ab|$.*

SOLUTION. Without loss of generality we may assume that a and b are positive. First we take $a > 0$, $b > 0$ with $(a, b) = 1$. Suppose $[a, b] = ma = nb$. Then b divides ma and $(b, a) = 1$ implies that b divides m. Therefore $b \le m$ and $ba \le ma = [a, b]$. But ba itself is a common multiple of a, b and hence $ba = [a, b]$. Thus $ab = ab.\ 1 = [a, b]\ (a, b)$. Now assume $a > 0$, $b > 0$, $(a, b) = d > 1$.

Then $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and hence by the first part of the proof

$$\left[\frac{a}{d}, \frac{b}{d}\right] = \frac{a}{d} \cdot \frac{b}{d}.$$

This gives $d^2\left[\frac{a}{d}, \frac{b}{d}\right] = ab$ or $d\left[d\frac{a}{d}, d\frac{b}{d}\right] = ab$

or $\qquad d[a, b] = ab$ i.e., $(a, b) \,.\, [a, b] = ab.$

EXAMPLE 12. *Let a, b, c be integers. The equation $ax + by = c$ has a solution if and only if (a, b) divides c. Also if (x_0, y_0) is a particular solution of $ax + by = c$ then a general solution is given by*

$$x = x_0 + t\frac{b}{(a,b)},\ y = y_0 - t\frac{a}{(a,b)} \quad \text{where} \quad t \in Z.$$

SOLUTION. Let $(a, b) = d$. It has been already proved in the Corollary to Theorem 3 that $ax + by = c$ has a solution if and only if $d \mid c$. Suppose x_0, y_0 is a particular solution and x_1, y_1 is any other solution. Then $ax_1 + by_1 = c = ax_0 + by_0$.

$$\therefore \qquad \frac{a}{d}(x_1 - x_0) = \frac{b}{d}(y_0 - y_1).$$

Now $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and $\frac{b}{d}$ divides $\frac{a}{d}(x_1 - x_0)$ implies that $\frac{b}{d}$ divides $x_1 - x_0$.

Hence there exists $t \in \mathbf{Z}$ such that $t\,\frac{b}{d} = x_1 - x_0$. This gives $\frac{a}{d}\ t\ \frac{b}{d} = \frac{b}{d}(y_0 - y_1)$ or

$t\frac{a}{d} = y_0 - y_1$. Thus $x_1 = x_0 + t\frac{b}{d}$ and $y_1 = y_0 - t\frac{a}{d}$.

EXAMPLE 13. *Find all the integral solution of $93x - 27y = 6$.*

SOLUTION. We have $(93, 27) = 3$; and 3 divides 6. Therefore there exist solutions. To find one particular solution x_0, y_0 we first apply the Euclidean algorithm to find a and b such that $a\ 93 + b\ 27 = 3$. We have

$$93 = 3 \cdot 27 + 12$$

$$27 = 2 \cdot 12 + 3$$
$$12 = 4 \cdot 3 + 0$$
$$3 = 1 \cdot 27 - 2 \cdot 12$$
$$= 1 \cdot 27 - 2\,(1 \cdot 93 - 3 \cdot 27)$$
$$= 7 \cdot 27 - 2 \cdot 93$$

Thus $\quad -2 \cdot 93 + 7 \cdot 27 = 3$

So $\quad -4 \cdot 93 + 14 \cdot 27 = 6$

In fact, $\quad -4 \cdot 93 - 14 \cdot (-27) = 6.$

Hence $x_0 = -4$ and $y_0 = -14$ constitute a particular solution. The general solution is now given by

$$x = -4 + k\left(\frac{-27}{3}\right), \quad y = -14 - k\left(\frac{93}{3}\right)$$

i.e., $\quad x = -4 - 9k, \quad y = -14 - 31k$

The table of different solutions may be given by

k	0	1	-1	...
x	-4	-13	5	...
y	-14	-45	17	...

EXAMPLE 14. *If $f(x)$ is a nonconstant polynomial with integral coefficients then f takes some composite values. (i.e., $f(x)$ cannot be a prime for all $x \in \mathbf{Z}$).*

SOLUTION. Let $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ with $a_i \in \mathbf{Z}$.

Suppose $f(k) = a_0 + a_1k + a_2k^2 + \ldots + a_nk^n = m \neq 1$ for some $k \in \mathbf{Z}$. Such a k exists since $f(x)$ is a nonconstant polynomial then

$$f(k+m) = a_0 + a_1\,(k+m) + a_2\,(k+m)^2 + \ldots + a_n\,(k+m)^n.$$

Expanding the terms of $f(k+m)$ we find that

$$f(k+m) = (\text{multiple of } m) + a_0 + a_1k + a_2k^2 + \ldots + a_nk^n$$
$$= (\text{multiple of } m) + f(k)$$
$$= (\text{multiple of } m) + m$$
$$= \text{multiple of } m = \text{a composite number.}$$

Thus there exist no nonconstant polynomial $f(x)$ with integral coefficients which takes only prime values.

EXERCISE 2.2

1. Find the greatest common divisors of the following pairs of integers
   (*a*) 537, 765 (*b*) 801, 423 (*c*) 12321, 8658 (*d*) 138, 1740.
2. Compute the *l.c.m.* of (*a*) 27, 30 (*b*) $n, n+1$ (*c*) 1234, 3702.
3. Given that $d_1 = \dfrac{d}{(b,d)}, \quad d_2 = \dfrac{d}{(b,d)}$

   show that $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad_1 + cd_2}{[b,d]}$.

4. Show that if a and b are nonzero integers

$$(a, b) \text{ divides } [a, b].$$

5. Prove that $(a + b, a - b) \geq (a, b)$ for any two integers.
6. Find integral values of x, y and z if

   (i) $243x + 198y = 9$ (ii) $61x - 40y = 1$

   (iii) $10x - 8y = 42$ (iv) $6x + 10y + 15z = -1$.
7. Prove that the product of any three consecutive integers is always divisible by 6.
8. Give an example of four integers which are relatively prime but not relatively prime in pairs.
9. Prove that any set of integers relatively prime in pairs is a set of relatively prime integers.
10. If $(a, 4) = 2$ and $(b, 4) = 2$ find $(a + b, 4)$.
11. For any n prove that $n^5 - n$ is divisible by 30.
12. Prove that if n is odd then $n^2 - 1$ is a multiple of 8.
13. Suppose $(a, b) = [a, b]$ for two integers, solve for a, b.
14. Find all integers n such that n^2 is of the form $3k + 2$.
15. Find all solutions of $(a, b) = 10$, $[a, b] = 100$.
16. Let $a > 0$ and $d > 0$ be two given integers. Prove that there exist integers x, y such that $(x, y) = d$, $xy = a$ if and only if d^2 divides a.
17. If $m > n$, prove that $a^{2^n} + 1$ is a divisor of $a^{2^m} - 1$. Find $(a^{2^m} + 1, a^{2^n} + 1)$ when a, m, n are positive integers and $m \neq n$.
18. If a is prime to b and y, b is prime to x then prove that $ax + by$ is prime to ab.
19. If $(a, b) = 1$ and n is a prime then prove that $(a^n + b^n)/(a + b)$ and $a + b$ have no common factors unless $a + b$ is a multiple of n.
20. If $m = a_1x + b_1y$, $n = a_2x + b_2y$ and $a_1b_2 - a_2b_1 = 1$ then prove that $(m, n) = (x, y)$.
21. If p is a prime and $a^2 - b^2 = p$ solve for a and b.

2.3 THE FUNDAMENTAL THEOREM OF ARITHMETIC

Every integer n can be factorised in at least one way namely $n = n \times 1$. If n is a prime integer then we see that the only factorisations possible are $n = n \times 1$ or $n = (-n)(-1)$. If n is a positive integer which is not a prime then we may write $n = ab$ with a, b being integers such that $1 < a, b < n$. Now consider $a > 1$. If a is not a prime then break it into factors as $a = a_1 a_2$ with $1 < a_1$ and $a_2 < a$. Similarly if b is not a prime, break it into factors as $b = b_1b_2$ with $1 < b_1, b_2 < b$. If we keep on doing this factorisation, ultimately we get $n = p_1 p_2 \dots p_k$, a product of k prime numbers. Thus any positive n can be written as a product of prime numbers. (Of course this requires a formal proof which we give later).

For example

$$\begin{aligned} 120 &= 2 \times 2 \times 2 \times 3 \times 5 \\ &= 2 \times 3 \times 2 \times 2 \times 5 \\ &= 2 \times 5 \times 3 \times 2 \times 2 \\ &= 2 \times 3 \times 5 \times 2 \times 2 \text{ etc.} \end{aligned}$$

We observe that in all these factorisations, the primes appearing are the same, although the order in which they appear are different. Is this observation true for prime factorisations of all the positive integers bigger than 1?

Theorem 6 (Fundamental Theorem of Arithmetic)

For any integer $n > 1$ there exist primes $p_1 \le p_2 \le ... \le p_k$ such that $n = p_1 p_2 ... p_k$. Furthermore, such a factorisation is unique.

Proof. First we shall prove that each positive integer $n > 1$ has a prime factorisation which has already been observed in the discussion preceding the theorem. This we prove by induction on n. Clearly it is true for $n = 2$. Assume that the result is true for all n such that $2 \le n \le k$. This is our induction hypothesis. Consider now $n = k + 1$. If $k + 1$ itself is a prime, there is nothing to prove; its prime factorisation will just consist of $(k + 1)$ itself. Otherwise we write $k + 1 = ab$ with $1 < a, b < k + 1$. Now $1 < a \le k$, $1 < b \le k$ and therefore by our induction hypothesis both a and b have prime factorisations. Let $a = p_1 p_2 ... p_m$ and $b = q_1 q_2 ... q_l$ be the prime factorisations of a and b respectively. Then $k + 1 = ab = p_1 p_2 ... p_m q_1 q_2, ...q_l$ is a Prime factorisation of $k + 1$. Thus, by the principle of mathematical induction, every integer $n > 1$ has a factorisation into primes.

Next we have to prove the uniqueness of prime factorisation. The uniqueness is true for $n = 2$. Assume the uniqueness of prime factorisation as stated in the theorem for $2 \le n \le k$. Suppose $k + 1 = p_1 p_2 ... p_m = q_1 q_2 ... q_l$ are two factorisations of $k + 1$ into primes such that $p_1 \le p_2 ... \le p_m$ and $q_1 \le q_2 \le ... \le q_l$. Since q_1 is a prime dividing the product $p_1 p_2 ... p_m$ we must have q_1 dividing p_i for some $i \in \{1, 2, ... m\}$. But p_i itself is a prime and therefore $q_1 = p_i$. Similarly $p_1 = q_j$ for some $j \in \{1, 2, ... l\}$. We have $p_1 = q_j \ge q_1 = p_i \ge p_1$. Therefore $p_1 = q_j = q_1 = p_i$ or $p_1 = q_1$. Now consider

$$A = \frac{k+1}{p_1} = p_2 p_3 ... p_m = q_2 q_3 ... q_l.$$

If $A = 1$, then $k + 1 = p_1 = q_1$ and the uniqueness is verified. If $A > 1$, then $1 < A < k + 1$ and by induction hypothesis A has a unique factorisation into primes in ascending order. This means that $m - 1 = l - 1$ and $p_2 = q_2, p_3 = q_3, ..., p_m = q_m$. This proves that $(k + 1)$ has a unique factorisation into primes in ascending order. Hence by the principle of mathematical induction, every integer $n > 1$ has a unique factorisation in the form $n = p_1 p_2 ... p_k$ where each p_i is a prime and $p_1 \le p_2 \le ... \le p_k$. This proves the Fundamental Theorem of Arithmetic. ❑

EXAMPLE 1. Let E be the set $\{2, 4, 6, 8, ...\}$ of positive even integers. We say that $x \in E$ is a prime in E if x is not the product of two elements of E. For example 14 is a prime in E. Here again, every element in E can be written as a product of primes. However, such a prime factorisation in E is not unique! For example $60 = 2 \times 30 = 6 \times 10$ are two different prime factorisations of 60 in E. This example shows that the mere existence of prime factorisation does not imply the uniqueness of such a factorisation.

EXAMPLE 2. *Show that the number of primes in* **N** *is infinite.*

SOLUTION. Suppose the number of primes in **N** is finite. Let $\{p_1, p_2 ..., p_n]$ be the set of primes in **N** such that $p_1 < p_2 < ... < p_n$. Consider $n = 1 + p_1 p_2 ... p_n$. Clearly n is not divisible by any one of $p_1, p_2, ..., p_n$. Hence n itself is a prime or n has a prime divisor other than $p_1, p_2 ..., p_n$. This contradicts that the set of primes is $\{p_1, p_2 ..., p_n\}$. Therefore the number of primes in **N** is infinite.

EXAMPLE 3. *Given any positive integer n, we can uniquely express n as a product of a non-negative power of 2 and an odd number.*

SOLUTION. Let $n = p_1^{a_1}\ p_2^{a_2}\ \dots\ p_k^{a_k}$ be the unique factorisation of n into primes with $p_1 < p_2 < \dots < p_k$. Then either $p_1 = 2$ or each $p_i > 2$ and hence odd. Therefore $n = 2^{a_1}\ b$ or $n = c$ where $b = p_2^{a_2}\ p_3^{a_3}\ \dots\ p_k^{a_k}$ and $c = p_1^{a_1}\ p_2^{a_2}\ \dots\ p_k^{a_k}$. When $n = 2^{a_1}\ b$, each p_i is odd for $i = 2, 3, \dots k$ and hence b is odd. Again when $n = c = p_1^{a_1}\ p_2^{a_2}\ \dots\ p_k^{a_k}$ each p_i is odd and hence c is odd. Thus $n = 2^{a_1}\ b$ or $n = 2^{\circ}.c$ = (a non-negative power of 2) × (an odd number). Uniqueness part is left as an exercise.

EXAMPLE 4. *Given any positive integer k, find k consecutive composite numbers.*

SOLUTION. Now all the positive integers $j \le k + 1$ divide $(k + 1)!$. Therefore $(k + 1)! + j$ is divisible by j for $j = 1, 2, 3, \dots, k + 1$. This gives k consecutive integers, viz., $2 + (k + 1)!, 3 + (k + 1)!, \dots, (k + 1) + (k + 1)!$ which are all composite.

EXAMPLE 5. *Use the unique factorisation theorem to find the l.c.m. and g.c.d. of 136 and 228.*

SOLUTION. We have

$$136 = 8 \times 17 = 2^3 \times 17 \text{ and}$$
$$228 = 2^2 \times 3 \times 19.$$
$$[136, 228] = 2^3 \times 3 \times 17 \times 19 = 7752 \text{ and}$$
$$(136, 228) = 2^2 = 4$$

If $\quad a = p_1^{a_1}\ p_2^{a_2}\ \dots\ p_k^{a_k}$ and

$b = p_1^{b_1}\ p_2^{b_2}\ \dots\ p_k^{b_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes, a_i and b_i are non-negative integers (one can always write two positive integers a and b in the above form) then

$$[a, b] = p_1^{\max(a_1, b_1)}\ p_2^{\max(a_2, b_2)} \dots p_k^{\max(a_k, b_k)}$$

$$(a, b) = p_1^{\min(a_1, b_1)}\ p_2^{\min(a_2, b_2)} \dots p_k^{\min(a_k, b_k)}.$$

EXAMPLE 6. *A positive integer n is a prime if (n, p) = 1 for every prime integer $p \le \sqrt{n}$. In other words a positive integer n is a prime if no prime $p \le \sqrt{n}$ divides n.*

SOLUTION. Let $(n, p) = 1$ for every prime $p \le \sqrt{n}$. Suppose n is not a prime, we may write $n = ab$ with $1 < a \le b$, then $a \le \sqrt{n}$. Any prime p dividing a also divides n and we have $p \le a \le \sqrt{n}$, contradicting our assumption on n. Hence n is a prime.

Note. The above Example 6 enables us to check whether a given number is prime or not. For example if one wants to check whether 187 is a prime, it is enough to check whether 2, 3, 5,7, 11, 13 divide 187, since these are the only primes less than or equal to $\sqrt{187}$.

EXAMPLE 7. *Let $n = p_1^{a_1}\ p_2^{a_2}\ \dots\ p_k^{a_k}$ be the unique factorisation of n into a product of distinct primes. Then the number of positive divisors of n is given by*

$$\tau(n) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \dots (a_k + 1).$$

SOLUTION. Further the sum of the divisors $\sigma(n)$ is given by

$$\sigma(n) = \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right)\dots\left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right).$$

Any positive divisor d of n must be of the form

$$d = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k} \text{ with } 0 \le b_i \le a_i \text{ for } i = 1, 2, \dots k.$$

Now consider the product

$$(1 + p_1 + p_1^2 + \dots + p_1^{a_1})(1 + p_2 + p_2^2 + \dots + p_2^{a_2}) \dots$$
$$(1 + p_k + p_k^2 + \dots + p_k^{a_k}) \quad (*)$$

Any typical term in this product is of the form

$$x = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k} \text{ with } 0 \le b_i \le a_i \text{ for } i \in \{1, 2, \dots k\}$$

This means that the number of positive divisors of n is the number of terms in the above product. This gives

$$\tau(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1).$$

[If we multiply $(x_1 + x_2 + \dots + x_m)$ with $(y_1 + y_2 + \dots + y_n)$ then a typical term is $x_i y_j$ and the number of such terms is mn. In the above product (*), i^{th} bracket $(1 + p_i + p_i^2 + \dots + p_i^{a_i})$ contains $a_i + 1$ terms. Hence the total number of terms in the product (*) is

$$(a_1 + 1)(a_2 + 1) \dots (a_k + 1)].$$

Again, using our earlier observation, we get

$\sigma(n)$ = the sum of positive divisors of n

$$= (1 + p_1 + p_1^2 + \dots + p_1^{a_1})(1 + p_2 + p_2^2 + \dots + p_2^{a_2}) \dots (1 + p_k + p_k^2 + \dots + p_k^{a_k})$$

$$= \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \dots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right)$$

(see Example 1 of Section 2.1 for the sums of the form $(1 + p_l + p_l^2 + \dots + p_l^{a_l})$.)

For example $\tau(60) = \tau(2^2 . 3^1 . 5^1) = (2 + 1)(1 + 1)(1 + 1) = 12$

$$\sigma(60) = \left(\frac{2^3 - 1}{2 - 1}\right)\left(\frac{3^2 - 1}{3 - 1}\right)\left(\frac{5^2 - 1}{5 - 1}\right) = 7.4.6 = 168$$

In fact the positive divisors of 60 are 1, 2, 3, 4, 5, 6, 10,12, 15, 20, 30, 60 which are 12 in number; and their sum = 168.

EXAMPLE 8. *Find the number of ways in which a positive integer n can be written as a product of two positive integers, including n and 1.*

SOLUTION. Case (i) $n = p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$ is a perfect square with the above prime factorisation. Then $\tau(n) = (2a_1 + 1)(2a_2 + 1)\dots(2a_k + 1)$ = an odd integer.

Let $1 = d_1 < d_2 < d_3 < \dots < d_{\tau(n)} = n$ be the distinct positive divisors of n in the ascending order. Suppose $\tau(n) = 2l - 1$, then the different factorisations are as follows $d_1 d_{\tau(n)}, d_2 d_{\tau(n) - 1}, \dots, d_r d_{\tau(n) - r + 1}, \dots. d_l d_l$.

There are l factorisations in all. We have $l = \dfrac{\tau(n) + 1}{2}$. Thus when n is a perfect square we have $\dfrac{\tau(n) + 1}{2}$ factorisations of n in the form $n = ab$,

$1 \le a \le b \le n.$

Case (ii) n is not a perfect square.

Let $n = p_1^{a_1}\, p_2^{a_2} \ldots p_k^{a_k}$ be the unique prime factorisation of n. Then at least one a_j is odd since n is not a perfect square. This implies that $\tau(n) = (a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$ is an even number.

If $1 = d_1 < d_2 < d_3 < \ldots < d_{\tau(n)} = n$ are the positive divisors of n then the distinct factorisations of n are $d_1\, d_{\tau(n)}$, $d_2 d_{\tau(n)-1}$, $d_3 d_{\tau(n)-2}$ ..., $d_l\, d_{l+1}$ where $\tau(n) = 2l$. Thus when n is not a perfect square we have $l = \dfrac{\tau(n)}{2}$ factorisations in the form $n = ab$.

For example if $n = 36 = 2^2 \times 3^2$, $\tau(n) = (2 + 1)(2 + 1) = 9$. The divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36. The distinct factorisations are 1×36, 2×18, 3×12, 4×9, 6×6 which are $5 = \dfrac{10}{2} = \dfrac{\tau(n)+1}{2}$ in number.

When $n = 60 = 2^2 \times 3 \times 5$, we have $\tau(n) = (2 + 1)(1 + 1)(1 + 1) = 12$. The divisors are 1, 2, 3, 4, 5, 6,10, 12, 15, 20, 30, 60 and the distinct factorisations are 1×60, 2×30, 3×20, 4×15, 5×12 and 6×10 which are $6 = \dfrac{12}{2} = \dfrac{\tau(n)}{2}$ in number.

EXAMPLE 9. *In Example 8 find the number of ways in which n can be written as a product of two factors, which are relatively prime to each other.*

SOLUTION. Let $n = p_1^{a_1}\, p_2^{a_2} \ldots p_k^{a_k}$ be the unique factorisation of n and $n = ab$ with $(a, b) = 1$. Then if p_j divides a then p_j does not divide b. This means that if P_j divides a then $P_j^{a_j}$ also divides a. Therefore any such factor of a or b must be a term in the expansion of $(1 + p_1^{a_1})(1 + p_2^{a_2}) \ldots (1 + p_k^{a_k})$ and vice versa. The number of terms in the above product is 2^k, and hence the number of factors a of n of the form $n = ab$ with $(a, b) = 1$ is 2^k. If $1 = d_1 < d_2 < d_3 < \ldots < d_{2^k} = n$ are these factors then the different factorisations of the required form are $d_1 d_{2^k}$, $d_2 d_{2^k - 1}$, ... $d_{2^{k-1}}\, d_{2^{k-1}+1}$ which are 2^{k-1} in number.

Thus, the number of ways in which n can be expressed as a product of two relatively prime factors is 2^{k-1}.

For example consider $n = 120 = 2^3 \times 3 \times 5$. Comparing with our calculations above, we have $k = 3$. Therefore, there are $2^{3-1} = 4$ ways in which 120 can be expressed in the desired form. They are 1×120, 3×40, 5×24, 8×15.

Definition 6. Define $[x]$ = integral part of x

= the greatest integer less than or equal to x for $x \in \mathbf{R}$.

For example $\left[\dfrac{3}{2}\right] = 1$, $[7.893] = 7$, $[\sqrt{2}] = 1$, $\left[\dfrac{-3}{2}\right] = -2$.

If x, y are integers and $x = qy + r$ with $0 \le r < y$ then $\left[\dfrac{x}{y}\right] = q$.

EXAMPLE 10. *If a_1, a_2, ... a_n are integers with $s = a_1 + a_2 + \ldots + a_n$ then*

$$\left[\frac{s}{a}\right] \ge \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \ldots + \left[\frac{a_n}{a}\right] \text{ for any integer } a > 0.$$

SOLUTION. We may write $a_j = aq_j + r_j$ with $0 \quad r_j \leq a$ for $j = 1, 2, \ldots n$ using division algorithm. Then $s = a_1 + a_2 + \ldots + a_n$ gives $s = a(q_1 + q_2 + \ldots + q_n) + (r_1 + r_2 + \ldots + r_n)$.

$$\therefore \quad \left[\frac{s}{a}\right] = \left[q_1 + q_2 + \cdots + q_n + \frac{(r_1 + r_2 + \cdots + r_n)}{a}\right]$$

$$\geq q_1 + q_2 + \cdots + q_n = \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \left[\frac{a_n}{a}\right]$$

EXAMPLE 11. *Find the largest power of a prime $p \leq n$, dividing $n!$.*

SOLUTION. We have $n! = 1 \times 2 \times 3 \times 4 \ldots. p \ldots 2p \ldots. p^2 \ldots. (p + 1)p \ldots. p^3 \ldots. n$. In the above product there are $[n/p]$ terms which are divisible by p, and among these there are $[n/p^2]$ terms which are divisible by p^2 and so on. Therefore, the highest power of p that divides $n!$ is

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^k}\right]$$

where p^k is the largest power of p which is less than or equal to n.

EXAMPLE 12. *Find the highest power of 5 that divides 518!*

SOLUTION. The highest power of 5 less than 518 is $125 = 5^3$. Therefore the highest power of 5 that divides 518! is

$$\left[\frac{518}{5}\right] + \left[\frac{518}{25}\right] + \left[\frac{518}{125}\right] = 103 + 20 + 4 = 127.$$

EXAMPLE 13. *Find the number of zeros that appear at the end in the representation of 158! in base 10.*

SOLUTION. If 10^k is the highest power of 10 that divides 158!, then we will have k zeros at the end of the decimal representation of 158! Now $10^k = 2^k \cdot 5^k$. The highest power of 2 that divides 158!

$$= \left[\frac{158}{2}\right] + \left[\frac{158}{4}\right] + \left[\frac{158}{8}\right] + \left[\frac{158}{16}\right] + \left[\frac{158}{32}\right] + \left[\frac{158}{64}\right] + \left[\frac{158}{128}\right]$$

$$= 79 + 39 + 19 + 9 + 4 + 2 + 1 = 153.$$

The highest power of 5 that divides 158! is given by

$$\left[\frac{158}{5}\right] + \left[\frac{158}{25}\right] + \left[\frac{158}{125}\right] = 31 + 6 + 1 = 38.$$

The highest power of 10 that divides 158! = min (153, 38) = 38.

Hence there are 38 zeros at the end of the decimal representation of 158!.

EXAMPLE 14. *Show that the product of any n consecutive integers is always divisible by n!.*

SOLUTION. Consider any such product $(k + 1)(k + 2) \ldots (k + n)$.

We have $$\frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{(k+n)!}{k!\,n!}$$

Let p be any prime divisor of $k!\,n!$. Then the highest power of p that divides $k!\,n!$ is

$$\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots + \left[\frac{k}{p^a}\right] + \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^b}\right]$$

where p^a is the highest power of p less than or equal to k and p^b is the highest power of p less than or equal to n. We may assume that $a \le b$. As seen in Example 10, we have

$$\left[\frac{k}{p}\right]+\left[\frac{n}{p}\right] \le \left[\frac{k+n}{p}\right], \left[\frac{k}{p^2}\right]+\left[\frac{n}{p^2}\right] \le \left[\frac{k+n}{p^2}\right] \text{ and so on.}$$

$$\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\cdots+\left[\frac{k}{p^a}\right]+\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots\left[\frac{n}{p^b}\right]$$

$$\le \left[\frac{k+n}{p}\right]+\left[\frac{k+n}{p^2}\right]+\cdots+\left[\frac{k+n}{p^a}\right]+\cdots\left[\frac{k+n}{p^c}\right]$$

where p^c is the highest power of p less than or equal to $(k + n)$.

This means that if p^x divides $n!\,k!$ then p^x divides $(k + n)!$

Therefore $\dfrac{(k+n)!}{k!\,n!}$ is an integer.

EXAMPLE 15. *Prove that there are infinitely many primes of the form 4k + 3 with k ∈ **Z**.*

SOLUTION. Let $n = 4k + 3 > 0$ and $\mathbf{S} = \{d > 0 \mid d$ is a divisor of n of the form $4m + 3$ with $m \in \mathbf{Z}\}$. Then $n \in \mathbf{S}$ and therefore $\mathbf{S} \neq \varphi$. We can speak of the least integer in $\mathbf{S}$ and let $p = \min\{d \mid d \in \mathbf{S}\}$. It is clear that $p > 1$. Suppose $p = ab$ with $1 < a, b < p$, then $p = 4m + 3$ is odd and hence both a and b must be odd integers. If $a = 4k_1 + 1$, $b = 4k_2 + 1$, then $ab = 4(4k_1k_2 + k_1 + k_2) + 1$. But ab is of the form $4m + 3$. Therefore, either a or b is of the form $4l + 3$. We may assume that $a = 4l + 3$. Then $a < p$ and a divides p implies that a divides n, contradicting the minimality of p. Hence p must be a prime. This means that every positive integer n of the form $4k + 3$ has a prime divisor of the same form. Now, for any $n \in \mathbf{N}$, $4n! - 1$ is of the form $4k + 3$ and hence $4n! - 1$ has a prime divisor p of the same form. But j does not divide $4n! - 1$ for $2 \le y \le n$. Therefore $p > n$. In other words, given any positive integer n, there exists a prime number p of the form $4k + 3$ with $p > n$. Hence there are infinitely many primes of the form $4k + 3$.

EXAMPLE 16. If x and y are prime numbers which satisfy $x^2 - 2y^2 = 1$, solve for x and y.

SOLUTION. $x^2 - 2y^2 = 1$ gives $x^2 - 2y^2 + 1$ and hence x must be an odd number. If $x = 2n + 1$, then $x^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 2y^2 + 1$. Therefore $y^2 = 2n(n + 1)$. This means that y^2 is even and hence y is an even integer. Now, y is also a prime implies that $y = 2$. This gives $x = 3$. Thus the only solution is $x = 3$, $y = 2$.

EXERCISE 2.3

1. An integer n whose prime factorisation is $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is a perfect square if and only if each a_i is even and n is a perfect cube if and only if each a_i is a multiple of 3; and in general n is a perfect mth power if and only if each a_i is a multiple of m.
2. Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, $n/5$ is a perfect fifth power.
3. Find all positive integers n such that $2^8 + 2^{11} + 2^n$ is a perfect square.
4. How many solutions are there in $\mathbf{N} \times \mathbf{N}$ to the equation $1/x + 1/y = 1/1995$?

5. If m, n, k are any three positive integers prove that
$(m, n)(m, k)(n, k)[m, n, k]^2 = [m, n][m, k][n, k](m, n, k)^2$.
6. Find the number of zeroes at the end of 1000!
7. Let $X = \{x \mid x = 1 + 1/2 + 1/3 + \ldots + 1/n, n \in \mathbf{N}\}$. Find $\mathbf{X} \cap \mathbf{N}$.
8. Find the smallest number with 28 divisors?
9. If a and b are coprime integers then prove that $((a + b)^m, (a - b)^m) \leq 2^m$ and $(a^n + b^n, a^n - b^n) \leq 2$.
10. Prove that there are infinitely many primes of the form $6n - 1$.
11. Show that $N = 101010\ldots101$ is not a prime except when N is 101.
12. Prove that there are infinitely many sets of five consecutive positive integers a, b, c, d, e such that $a + b + c + d + e$ is a perfect cube and $b + c + d$ is a perfect square.
13. Let $A = \{n \in \mathbf{N} \mid n$ is the sum of seven consecutive integers $\}$
$B = \{n \in \mathbf{N} \mid n$ is the sum of eight consecutive integers $\}$
$C = \{n \in \mathbf{N} \mid n$ is the sum of nine consecutive integers $\}$
Find $A \cap B \cap C$.
14. If 'a' is not a multiple of a prime p, then prove that there is an integer b such that $p^b - 1$ is a multiple of a.

PROBLEMS

1. Prove that
$S = 1 - 2^2 + 3^2 - 4^2 + 5^2 \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} n(n + 1)/2$.
2. Prove that $1.1! + 2.2! + 3.3! + \ldots.. + n.n! = (n + 1)! - 1$.
3. Prove that for any positive integer $n > 1$,
$$1/(n + 1) + 1/(n + 2) + \ldots. + 1/2n > 13/24.$$
4. Use induction to prove that
$$1 + 1\sqrt{2 + 1/\sqrt{3}} + \ldots. + 1/\sqrt{n} < 2\sqrt{n}.$$
5. Use induction to prove that $2!4!\ldots(2n)! > ((n + 1)!)^n$.
6. Prove that in any party the number of people who have made an odd number of handshakes is always even.
7. Prove that every positive integer having 3^m equal digits is divisible by 3^m.
8. $(n + 1)$ numbers are picked at random from the $2n$ integers $1, 2 \ldots. 2n$. Prove that among the numbers picked we can find at least two, one of which is divisible by the other.
9. The plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to colour each of these regions red or green in such a way that no two adjacent regions have the same colour.
10. If each person in a group of n people is a friend of at least half the people in the group, then prove that it is possible to seat them in a circle so that every one sits next to a friend of his/hers.
11. Prove that for a positive integer n,
$$11^{n+2} + 12^{2n+1} \text{ is divisible by } 133.$$
12. Prove by induction that $\sqrt{2}$ is irrational.
13. Prove that $x^2 + y^2 = z^n$ has a solution in $\mathbf{N}$, for all $n \in \mathbf{N}$.
14. Assuming that for every integer $n > 1$ there is a prime number between n and $2n$, prove that every positive integer can be written as a sum of distinct primes. (For this problem we treat 1 as a prime.)

15. A positive integer decreases an integral number of times when its last digit is deleted. Find all such numbers.

16. If the leading digit of a positive integer is deleted, the number gets reduced by 57 times. Find all such numbers, Find all positive integers such that if the leading digit is deleted the number gets reduced by 58 times.

17. Show that $3x^{10} - y^{10} = 1991$ has no integral solutions.

18. Find n if $2^{200} - 31.2^{192} + 2^n$ is a perfect square.

19. If a and b are integers and 3 divides $a^2 + b^2$, show that 3 divides a and b.

20. Find all integral solutions of $x^4 + y^4 + z^4 - w^4 = 1995$.

21. A positive integer gets reduced by nine times when one of its digits is deleted and the resultant number is divisible by 9. Prove that to divide the resultant number by 9, it is again sufficient to delete one of its digits. Find all such numbers.

22. Let n be a positive integer and m be a number having the same digits as that of n, but arranged in some other order. Prove that if $n + m = 10^{10}$, then n is divisible by 10.

23. Prove that for any integer n we have $n^k - n$ is divisible by k, where $k = 3, 5, 7, 11, 13$.

24. Prove that $5^{6n} - 3^{6n}$ is divisible by 152 for $n \in \mathbf{N}$.

25. Prove that for any two integers a and b $ab(a^{60} - b^{60})$ is divisible by 56786730.

26. Prove that $n!/a!b! \ldots k!$ is an integer if $a + b + \ldots k \leq n$.

27. Prove that in $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

28. Prove that for any positive integer n, $1^n + 2^n + 3^n + 4^n$ is divisible by five, if and only if, n is not divisible by 4.

29. Consider $S = \{a, a + d, a + 2d, \ldots\}$ where a and d are positive integers. Show that there are infinitely many composite numbers in S.

30. Prove that for any positive integers m and n, there exist a set of n consecutive positive integers each of which is divisible by a number of the form a^m, where a is some integer in N.

CHAPTER

3 GEOMETRY—STRAIGHT LINES AND TRIANGLES

3.1 STRAIGHT LINES

In the mathematical development of any branch of science, each definition of a general notion or concept involves other notions and relations. To avoid this logical hurdle we allow certain primitive concepts and relations as undefined concepts and relations. In fact all our theorems are of the form "Given that A is true it is implied that B is true". We have a hypothesis and we try to prove our proposition using certain axioms involving the undefined primitive concepts. Euclid, when he developed his geometry, did not clearly state what his primitive concepts were; but built his geometry based on certain axioms and postulates. Some of his primitive concepts were (1) the notion of a point, (2) the idea of intermediacy (*i.e.*, between any two points on a straight line there is another point. Compare this with the completeness axiom of the real numbers), (3) the idea of congruence. We do not intend to give any logical introduction to the Foundations of Geometry. Instead we give a reasonably systematic development of Euclidean Geometry and go on to prove some of the beautiful theorems of Euclid's Geometry. We fix a plane and study the geometry of straight lines, triangles and other geometric objects lying on this plane. Hereafter, all our geometric objects, unless otherwise stated, lie on this fixed plane.

By a line segment AB we mean the geometric figure consisting of two points A and B on a straight line and all the points lying between them. It is denoted by AB. We note that if three points A, B and C are taken on a straight line, exactly one of A, B, C lies between the other two. Also, a straight line l divides a plane into two disjoint half planes H_1 and H_2 such that the plane is the disjoint union of H_1, H_2 and l. If the ends of a line segment AB both belong to the same half plane, then AB does not intersect the line l. If the ends A and B are in different half planes demarkated by l then AB intersects the straight line l. We use the following notations:

$\overleftrightarrow{AB}$ – for the straight line determined by A and B

$\overrightarrow{AB}$ – for the ray AB with vertex at A

$\overline{AB}$ – for the line segment AB

AB – for the length of the line segment AB.

A point A on a straight line l (Fig. 1) divides the straight line into two rays, namely $\overrightarrow{Al_1}$ and $\overrightarrow{Al_2}$. (Note that $\overrightarrow{Al_1}$ is not a line segment, it is a half line with A as its initial point).

The rays of the straight line l into which it is broken by the point A will be called *supplementary* rays. In Fig. 3.1, the rays $\overrightarrow{Al_1}$ and $\overrightarrow{Al_2}$ are supplementary rays.

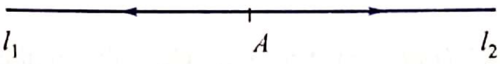


Fig. 3.1

For any three points A, B and C we have the distance $|AC| \leq$ distance $|AB|$ + distance $|BC|$. Here distance $|AB|$ stands for the distance between the points A and B. Two distinct points determine a straight line and two distinct straight lines have either one common point or none.

An angle is a figure which consists of two different rays with a common origin, as in Fig. 3.2. For the angle AOB in Fig. 3.2, O is the vertex of the angle and the rays $\overrightarrow{OA}, \overrightarrow{OB}$ are known as the sides of the angle AOB.

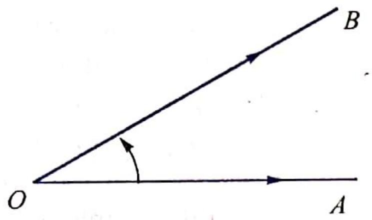


Fig. 3.2

If the sides OA, OB are supplementary then the angle AOB is a straight angle and its measure is taken to be 180 degrees. We write angle AOB as $\angle AOB$. Other units in measuring angles are minutes and seconds. We have 60 minutes = 1 degree and 60 seconds = 1 minute. We use the symbols $\theta°$, θ', θ'' to mean θ degrees, θ minutes and θ seconds respectively.

While measuring lengths of line segments and angles we observe the following fundamental principles or rules.

(1) Every line segment has a positive length.

(2) If a point C on a straight line l lies between the points A and B on l, then the length of the line segment AB is equal to the sum of the lengths of the line segments AC and CB.

(3) Every angle has a certain magnitude. The measure of a straight angle is 180°.

(4) If a ray OC lies between the sides of the angle AOB as in Fig. 3.3, then $\angle AOB = \angle AOC + \angle COB$.

(5) In measuring the angles we follow the convention, namely the angles which are anticlockwise like $\angle AOB$, $\angle AOC$ and $\angle COB$ of Fig. 3.3, are taken to be positive; and the angles BOA, COA, BOC of Fig. 3.4, which are clockwise are taken to be negative.

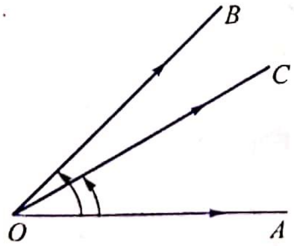


Fig. 3.3

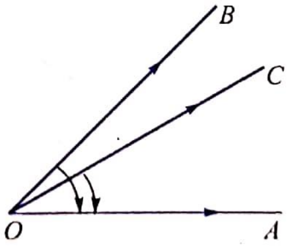


Fig. 3.4

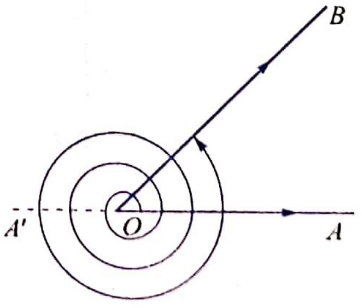


Fig. 3.5

(6) If a ray $\overrightarrow{OA}$ revolves around the point **O** (in our fixed plane, of course) k times in the positive direction and reaches the position. $\overrightarrow{OB}$, then the angle described is $k\,(360°) + \angle AOB°$ (Fig. 3.5). Note that when $\overrightarrow{OA}$ makes a half turn and reaches $\overrightarrow{OA'}$, by our definition the angle turned is one straight angle or 180°; and therefore when $\overrightarrow{OA}$ makes one complete revolution and comes back to its original position, it has described two straight angles or 360°. When the rotation is clockwise, we take the corresponding angles to be negative.

An angle whose measure is 90° or half the measure of a straight angle is called a *right angle.* If the angle between two rays is zero, then the two rays are coincident. Conversely if the two rays are coincident, then the angle between them is an integral multiple of 360°.

EXAMPLE 1. (*i*) $\angle AOA' = 180°$ in the positive sense and

$\angle AOA' = -\,180°$ in the negative sense (Fig. 3.6)

(*ii*) $\angle AOB = \angle BOA' = \frac{1}{2}(180°) = 90°$ and $\angle BOA = -\,90°$. In general $\angle XOY = -\,\angle YOX$ unless $\angle XOY$ is a straight angle, in which case it depends on the sense of description (see (*i*))

(*iii*) $\angle AOB = 30°$ (Fig. 3.8)

(*iv*) $\angle AOB = 150°$ (Fig. 3.9)

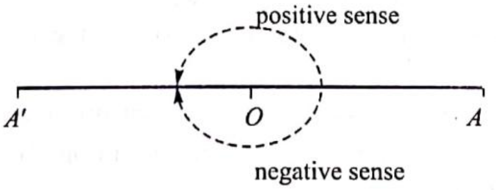


Fig. 3.6

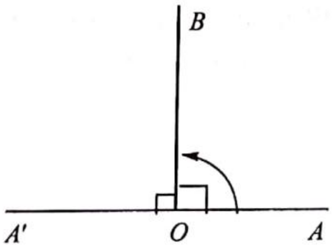


Fig. 3.7

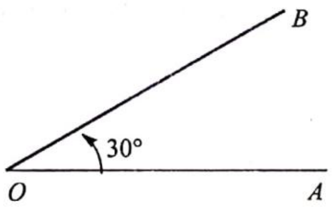


Fig. 3.8

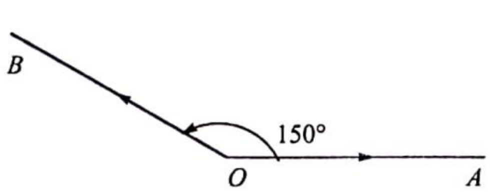


Fig. 3.9

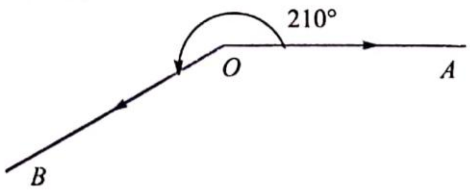


Fig. 3.10

(v) $\angle AOB = 210^\circ$ (Fig. 3.10)

(vi) Let the plane be divided into four quadrants by the two straight lines $X'OX$ and $Y'OY$ such that $\angle XOY = \angle YOX' = \angle X'OY' = \angle Y'OX = 90^\circ$ = a right angle, as in Fig. 3.11. Then any ray $\overrightarrow{OA_1}$ in the first quadrant makes an angle XOA_1 with OX such that $0 < XOA_1 < 90^\circ$. Any ray $\overrightarrow{OA_2}$ in the second quadrant makes an angle XOA_2 with OX such that $90^\circ \angle XOA_2 < 180^\circ$; any ray OA_3 in the third quadrant makes an angle XOA_3 with OX such that $180^\circ < \angle XOA_3 < 270^\circ$; any ray $\overrightarrow{OA_4}$ in the fourth quadrant makes an angle XOA_4 with OX such that $270^\circ < \angle XOA_4 < 360^\circ$ (note the sense of description of the

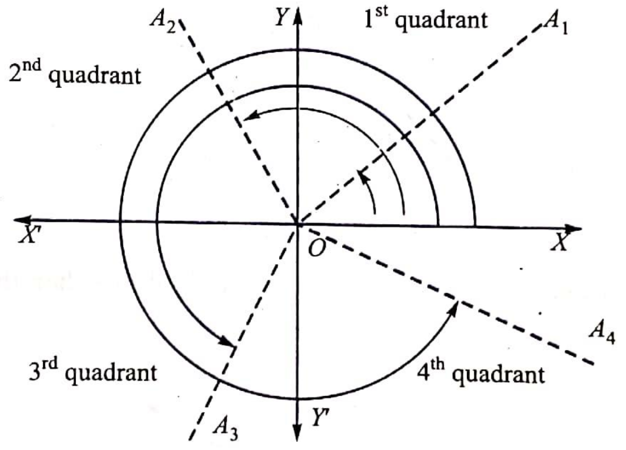


Fig. 3.11

angles). In general any ray $\overrightarrow{OA}$ in the first quadrant makes an angle $XOA = \theta$ with $\overrightarrow{OX}$ such that $(360\,k)^\circ < \theta^\circ < (360\,k + 90)^\circ$ for some nonnegative integer k. For example it could be $360^\circ < \theta^\circ < (360^\circ + 90^\circ)$ or $720^\circ < \theta^\circ < (720^\circ + 90^\circ)$ as in Fig. 3.12(*a*) or Fig. 3.12(*b*).

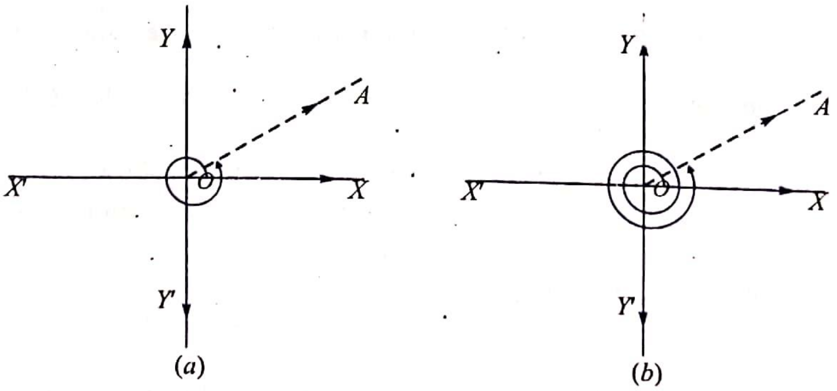


Fig. 3.12

Angles θ for which $0 < \theta° < 90°$ are called acute angles. Angles for which $90° < \theta° < 180°$ are called obtuse angles. Two angles α, β with $0 \le \alpha°, \beta° \le 180°$ are supplementary if $\alpha + \beta = 180°$; are complementary if $\alpha + \beta = 90°$.

Definition 1. If two angles AOB and BOC have a common vertex O, a common arm $\overrightarrow{OB}$ and if they are situated on the opposite sides of their common arm $\overrightarrow{OB}$, then they are called *adjacent angles*.

$\angle AOB$ and $\angle BOC$ are adjacent angles in Fig. 3.13.

Note. Recall that all our geometric objects lie on a fixed plane. This standing assumption also includes the geometric objects appearing in definitions.

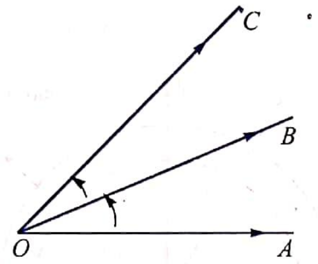


Fig. 3.13

Theorem 1. If a straight line stands on another straight line then the sum of the two adjacent angles is two right angles.

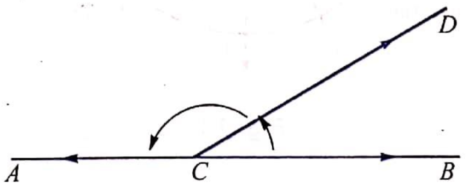


Fig. 3.14

Proof. The straight line CD stands on the straight line $\overleftrightarrow{ACB}$. It is required to prove that the sum of the adjacent angles BCD and DCA is 180°. By hypothesis, $\overleftrightarrow{ACB}$ is a straight line and hence $BCA = 180°$; and the rays $\overrightarrow{CB}, \overrightarrow{CA}$ are supplementary rays. The ray $\overrightarrow{CD}$ lies between the sides $\overrightarrow{CB}, \overrightarrow{CA}$ of the straight angle BCA and hence $\angle BCD + \angle DCA = 180°$. ❑

The converse of Theorem 1 is also true, which we state as follows.

Theorem 2. If the sum of two adjacent angles AOC and COB with the common arm $\overrightarrow{OC}$ is two right angles, then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are supplementary rays.

Proof. We are given that $\angle AOC + \angle COB = 2$ right angles $= 180°$. It is required to prove that BOA is a straight line. Now, extend the line segment $\overrightarrow{BO}$ to D such that BOD is a straight line. By construction BOD is a straight line and the straight line OC stands on it. Therefore by Theorem 1, $\angle DOC + \angle COB = 180°$. Also, by our hypothesis

$\angle AOC + \angle COB = 180°$. This gives $\angle AOC + \angle COB = \angle DOC + \angle COB$ or $\angle AOC = \angle DOC$. This means that the ray $\overrightarrow{OD}$ coincides with the ray $\overrightarrow{OA}$. In other words, BOA is a straight line. ❑

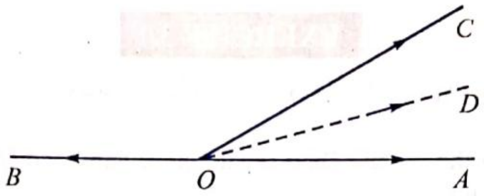


Fig. 3.15

Note. When we write $\angle AOC$ we follow our convention that anticlockwise angles are positive. Hence a possibility like Fig. 3.16 is ruled out.

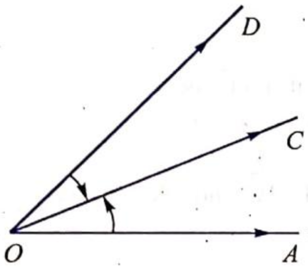


Fig. 3.16

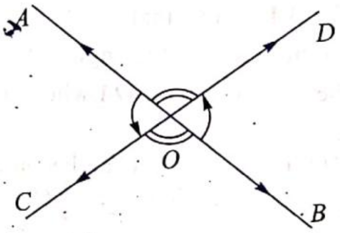


Fig. 3.17

Definition 2. Two angles are *vertically opposite angles* if the sides of one of them are supplementary rays of the sides of the other.

In Figure 3.17, $\angle BOD$ and $\angle AOC$ are vertically opposite angles. Also $\angle DOA$ and $\angle COB$ are vertically opposite angles.

Theorem 3. If two straight lines intersect the vertically opposite angles so formed are equal.

Proof. The straight lines AOB and COD intersect at the point O. $\angle BOD$ and $\angle AOC$ is one pair of vertically opposite angles so formed; and the other pair is $\angle DOA$ and $\angle COB$. It is required to prove that $\angle BOD = \angle AOC$ and $\angle DOA = \angle COB$. Now, AOB is a straight line and the straight line OD stands on it. Therefore by Theorem 1,

$$\angle BOD + \angle DOA = 180°. \quad (1)$$

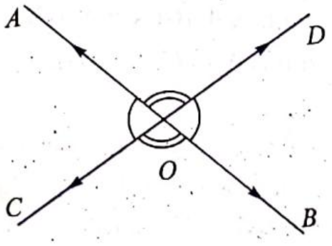


Fig. 3.18

Again by Theorem 1, applied to the line COD and the straight line OA standing on it we get

$$\angle DOA + \angle AOC = 180° \quad (2)$$

From (1) and (2) we see that $\angle BOD = \angle AOC$.

Similarly, we observe that $\angle DOA = \angle COB$.

Hence the theorem. ❑

EXERCISE 3.1

1. What is the angle in degrees between the hands of a watch at (*i*) 4 O' clock (*ii*) 5 hrs and 45 mts.
2. What angles do (*i*) the minute hand (*ii*) the hour hand and (*iii*) the seconds hand turn through in 20 minutes.
3. A straight line segment $\overline{AB}$ is bisected at C and produced to D. Show that $AD + BD = 2CD$.
4. A straight line segment $\overline{AB}$ is bisected at C and D is any point on CB. Prove that $AD - DB = 2CD$.
5. In Fig. 3.17 prove that
   (*i*) the bisectors of the angles DOA and DOB are at right angles.
   (*ii*) the bisector of $\angle DOA$ when produced also bisects $\angle COB$.
6. $\angle XOA$ and $\angle XOB$ are angles on the same side of $\overrightarrow{OX}$ and $\overrightarrow{OC}$ bisects $\angle AOB$. Prove that $\angle XOA + \angle XOB = 2\angle XOC$.
7. AOX, XOB are adjacent angles, in which $\angle AOX > \angle XOB$; $\overrightarrow{OC}$ bisects $\angle AOB$. Prove that $\angle AOX - \angle XOB = 2\angle COX$. (Compare the problems 3, 4 with problems 6,7)
8. If the bisectors of two adjacent angles are perpendicular to one another, then prove that the adjacent angles are formed by two intersecting straight lines.

3.2 CONGRUENCE OF TRIANGLES

Definition 3. If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, then the two triangles are *congruent*.

In other words two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent if $A_1B_1 = A_2B_2$, $B_1C_1 = B_2C_2$, $C_1A_1 = C_2A_2$, $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$ and $\angle C_1 = \angle C_2$. We write $\Delta A_1B_1C_1 \equiv A_2B_2C_2$ to mean that the two triangles are congruent. We observe that

(1) $\Delta ABC \equiv \Delta ABC$ for any triangle ABC

(2) If $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$, then $\Delta A_2B_2C_2 \equiv \Delta A_1B_1C_1$

(3) If $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$ and $\Delta A_2B_2C_2 \equiv \Delta A_3B_3C_3$ then $\Delta A_1B_1C_1 \equiv \Delta A_3B_3C_3$.

We take the following test for congruence of two triangles as an axiom.

SAS test: Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.

This says that, if in two triangles, $A_1B_1C_1$ and $A_2B_2C_2$ $A_1B_1 = A_2B_2$, $A_1C_1 = A_2C_2$ and $\angle A_1 = \angle A_2$, then the two triangles are congruent. This is known as the "*Side Angle Side*" test.

Theorem 4. The sum of any two angles of a triangle is less than a straight angle.

Proof. Let D be the mid point of the side BC of a triangle ABC. Produce AD to E such that $AD = DE$ (Fig. 3.19). Then by the SAS test the triangles ADB and EDC are congruent. Therefore, $\angle ABC = \angle ECD$ and hence $\angle ABC + \angle BCA = \angle ECD + \angle BCA = \angle ECA < 180°$ (note that E cannot lie on AC since the two distinct lines AD and AC have only one common point, namely A). Thus, the sum $\angle ABC + \angle BCA$ is less than two right angles. Similar arguments show that $\angle C + \angle A$ and $\angle A + \angle B$ are also each less than 180°. ❑

Corollary 1. Any exterior angle of a triangle is greater than any of the two non-adjacent interior angles.

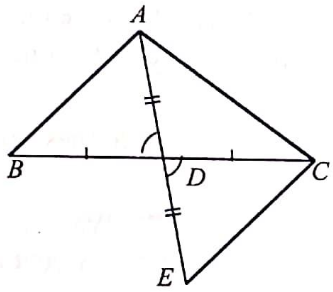


Fig. 3.19

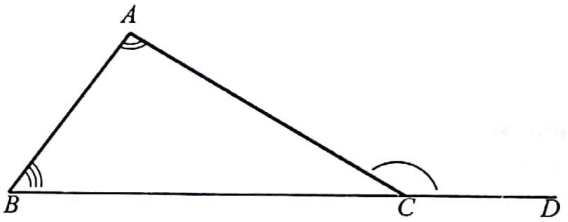


Fig. 3.20

Proof. Let ABC be any triangle and D be a point on BC produced as shown in Fig. 3.20. We want to prove that the exterior angle DCA is bigger than each of the non-adjacent interior angles A and B. By Theorem 1, $\angle DCA + \angle ACB = 180°$. By the theorem (Theorem 4) $\angle BCA + \angle CAB < 180° = \angle DCA + \angle BCA$. Therefore $\angle CAB < \angle DCA$. Again $\angle ABC + \angle BCA < 180° = \angle DCA + \angle BCA$ implies that $\angle ABC < \angle DCA$. Thus the exterior angle DCA is bigger than each of $\angle A$ and $\angle B$ of ΔABC. ❑

Corollary 2. In any triangle ABC, at most one of the angles A, B, C can be obtuse.

Proof. Immediate from Theorem 4. ❑

Theorem 5. (ASA theorem)

Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other.

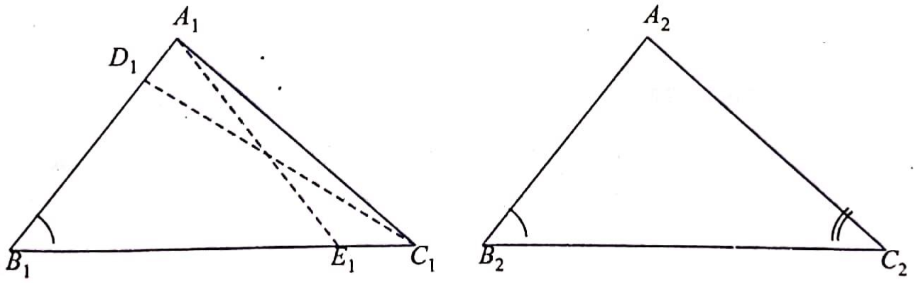


Fig. 3.21

Proof. Case (*i*) Suppose in triangles $A_1B_1C_1$ and $A_2B_2C_2$ we have $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and $B_1C_1 = B_2C_2$. Take D_1 on B_1A_1 such that $B_1D_1 = B_2A_2$ (we may assume without loss of generality $A_2B_2 \leq A_1B_1$) (Fig. 3.21). Then in the two triangles $B_1C_1D_1$ and $B_2C_2A_2$,

we have, the two sides B_1C_1 and B_1D_1 of $\Delta B_1C_1D_1$ equal in length to the two sides B_2C_2 and B_2A_2 respectively of $\Delta B_2C_2A_2$.Further the included angles B_1 and B_2 are equal. Therefore by *SAS* test, the two triangles are congruent. This means that $\angle B_1C_1D_1 = \angle B_2C_2A_2$. But $\angle B_2C_2A_2 = \angle B_1C_1A_1$ by our assumption. Therefore $\angle B_1C_1D_1 = \angle B_1C_1A_1$ and hence by our fundamental principle of measuring angles, the ray C_1D_1 coincides with the ray C_1A_1. This in turn implies D_1 coincides with A_1. Thus $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$. ❑

Case (*ii*) $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and the side A_1B_1 of $\Delta A_1B_1C_1$ = side A_2B_2 of $\Delta A_2B_2C_2$. Take E_1 on B_1C_1 such that $B_1E_1 = B_2C_2$ (as in Fig. 3.21; again without loss of generality we may assume that $B_1C_1 > B_2C_2$). Then the two triangles $B_1E_1A_1$ and $B_2C_2A_2$ are congruent by the *SAS* test. Now, for $\Delta A_1E_1C_1$, the exterior angle $A_1E_1B_1$ = the interior angle $A_1C_1E_1$ which is against the corollary 1 to Theorem 4 unless the point E_1 concides with C_1 in which case $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$. ❑

Theorem 6. If two sides of a triangle are equal then the angles opposite to these sides are equal.

Proof. In $\angle ABC$ let $AB = AC$. Compare the two triangles ABC and ACB. We have $AB = AC$, $AC = AB$ and $\angle BAC = \angle CAB$. Therefore by the *SAS* test $\Delta ABC \equiv \Delta ACB$ and hence $\angle ABC = \angle ACB$.

Aliter. Suppose AD bisects $\angle BAC$ (Fig. 3.22) meeting BC at D. In the triangles ABD and ACD, we may apply the *SAS* test to get $\Delta ABD \equiv \Delta ACD$. Therefore $\angle ABD = \angle ACD$ or $\angle B = \angle C$ in ΔABC. ❑

Definition 4. A triangle in which two sides are equal is an *isosceles triangle.*

Theorem 7. If in a triangle two angles are equal, then it is isosceles.

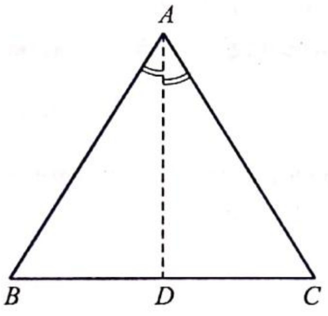


Fig. 3.22

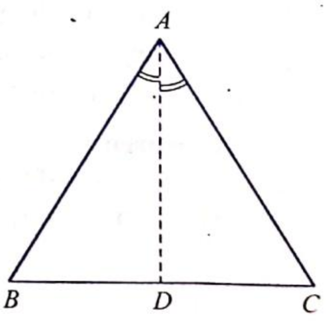


Fig. 3.23

Proof. Let $\angle ACB = \angle ABC$ in ΔABC. Compare the two triangles BCA and CBA. We have $BC = CB$, $\angle BCA = \angle CBA$ (assumption) and $\angle BAC = \angle CAB$ (common angle). Therefore by the *ASA* theorem $\Delta\ BCA \equiv \Delta\ CBA$ and hence $AB = AC$.

Aliter. Let AD bisect $\angle BAC$ meeting BC at D (Fig. 3.23). In the triangles ABD and ACD, $\angle ABD = \angle ACD$ (hypothesis), $\angle BAD = \angle CAD$ (construction), $AD = AD$ (common side). Therefore by the *ASA* theorem, $\Delta ABD \equiv \Delta ACD$ and hence we have $AB = AC$. ❑

Corollary. In an isosceles triangle, the median to the base bisects the vertical angle and further, is perpendicular to the base.

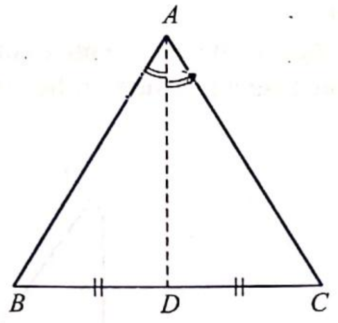


Fig. 3.24

Proof. Let ABC be an isosceles triangle with $AB = AC$ and let AD be the median to the base (the line joining a vertex of a triangle to the midpoint of the opposite side is called a median). In triangles ABD and ACD, we have $AB = AC$, $BD = CD$ and $\angle ABD = \angle ACD$ (in view of Theorem 6). Therefore, by the SAS test $\Delta ABD \equiv \Delta ACD$ and hence $\angle BAD = \angle CAD$ or AD bisects the vertical angle A. Now $\Delta ABD \equiv \Delta ACD$ also implies that $\angle ADB = \angle ADC$ and therefore by Theorem 1, each one of them must be a right angle. In other words the median $AD \perp BC$. ❑

Theorem 8. (SSS Theorem)

If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.

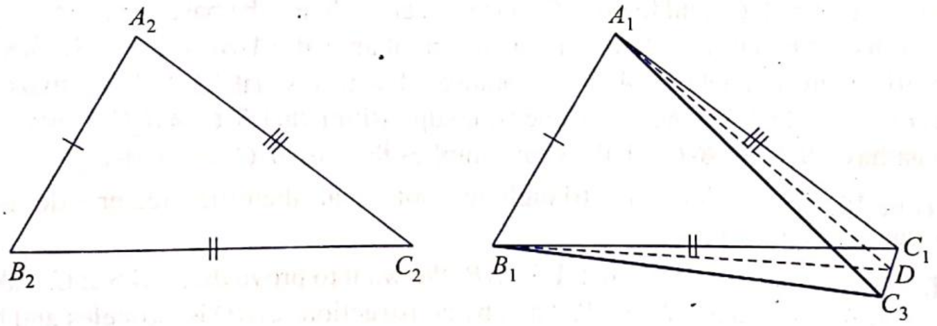


Fig. 3.25

Proof. In triangles $A_1B_1C_1$ and $A_2B_2C_2$ we are given that $A_1B_1 = A_2B_2$, $B_1C_1 = B_2C_2$ and $C_1A_1 = C_2A_2$. Suppose $\angle A_1 = \angle A_2$ or $\angle B_1 = \angle B_2$ then by our fundamental SAS test the two triangles are congruent. Otherwise assume that $\angle A_1 \neq \angle A_2$ and $\angle B_1 \neq \angle B_2$. Draw AC_3 as in Fig. 3.25 such that $\angle B_1A_1C_3 = \angle B_2A_2C_2$ and C_3, C_1 lie on the same side of the side A_1B_1, and further such that $A_1C_3 = A_2C_2$. Now, $A_1C_1 = A_2C_2$ by assumption. Therefore $\Delta A_1C_3C_1$ is isosceles. Again in triangles $A_1B_1C_3$ and $A_2B_2C_2$ we have $A_1B_1 = A_2B_2$, $A_1C_3 = A_2C_2$ and $\angle B_1A_1C_3 = \angle B_2A_2C_2$ (by construction). Applying the SAS test we note that $\Delta A_1B_1C_3 \equiv \Delta A_2B_2C_2$. Therefore $B_1C_3 = B_2C_2$; also $B_1C_1 = B_2C_2$ by hypothesis. This implies that $\Delta B_1C_1C_3$ is isosceles. If D is the mid point of C_1C_3 then A_1D and B_1D are both $\perp C_1C_3$ (Cor. to Theorem 7). Now, the straight lines A_1D and B_1D are distinct since D lies on the segment C_1C_3 and C_1, C_3 are both on the same side of the straight line A_1B_1. This means that we have two distinct perpendiculars DA_1, DB_1 through the point D on C_1C_3 to the sttaight line C_1C_3, which is impossible. Our assumption $\angle A_1 \neq \angle A_2$ and $\angle B_1 \neq \angle B_2$ has led to this impossibility. Therefore $\angle A_1 = \angle A_2$ or $\angle B_1 = \angle B_2$ and in either case we have already observed that the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent. ❑

Theorem 9. (RHS Theorem)

If in two right angled triangles, the hypotenuse and a side of one triangle are respectively equal to the hypotenuse and a side of the other, then the two triangles are congruent.

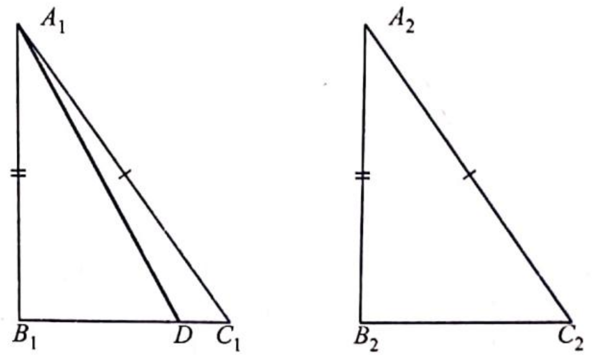


Fig. 3.26

Proof. In the triangles $A_1B_1C_1$ and $A_2B_2C_2$ we are given that $\angle B_1 = \angle B_2 = 90°$, the hypotenuse A_1C_1 = the hypotenuse A_2C_2 and $A_1B_1 = A_2B_2$. If $B_1C_1 = B_2C_2$ then by SSS Theorem, the two triangles are congruent. Suppose $B_1C_1 \neq B_2C_2$. We may assume $B_1C_1 > B_2C_2$. Take D on B_1C_1 such that $B_1D = B_2C_2$. Then by construction $\Delta A_1B_1D \equiv \Delta A_2C_2B_2$ (SAS test). This implies that $A_1D = A_2C_2$; but by hypothesis $A_1C_1 = A_2C_2$. Therefore $A_1D = A_1C_1$ and the ΔA_1C_1D is isosceles. Hence the base angles A_1C_1D and A_1DC_1 must be equal. By Theorem 4, in the right angled ΔA_1B_1D, $\angle A_1DB_1$, has to be necessarly acute and hence $\angle A_1DC_1$ is obtuse. This means that ΔA_1C_1D has two obtuse angles contradicting Theorem 4. Hence our supposition that $B_1C_1 \neq B_2C_2$ is wrong and we must have $B_1C_1 = B_2C_2$. This is turn implies that $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$. ❑

Theorem 10. If two sides of a triangle are not equal, then the greater side has the greater angle opposite to it.

Proof. In ΔABC we are given that $AC > AB$. We want to prove that $\angle B > \angle C$. Take the point D on AC such that $AD = AB$. Then by construction ΔABD is isosceles and hence $\angle ABD = \angle ADB$. By the Cor. 1. to Theorem 4, the exterior angle $BDA > \angle ACB$. Therefore $\angle ABD > \angle ACB$. Clearly, $\angle ABC > \angle ABD$ (Fig. 3.27). Thus $\angle ABC > \angle ABD > \angle ACB$ or $\angle B > \angle C$ whenever $AC > AB$. ❑

Theorem 11. If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Proof. In ΔABC we are given that $\angle B > \angle C$. (Fig. 3.28). We want to prove that $AC > AB$. Since $\angle B > \angle C$, we note that $AB \neq AC$. If $AC < AB$, then by Theorem 10 we must have $\angle C > \angle B$, which is against our hypothesis. Therefore $AC > AB$. ❑

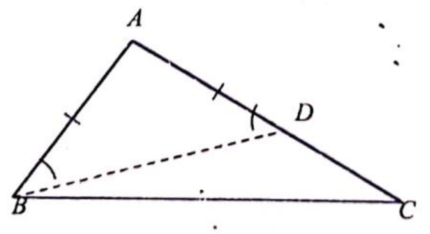


Fig. 3.27

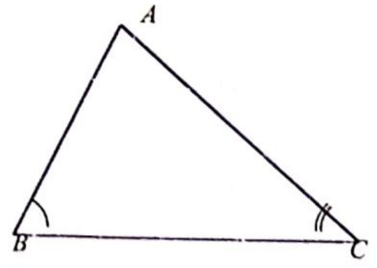


Fig. 3.28

Remark. In a right angled triangle, the two angles other than the right angle are acute and hence the hypotenuse is the largest side. This implies that of all the straight line segments drawn from a given point A not lying on a straight line l to meet l, the perpendicular has the least length.

Let D be the foot of the perpendicular from A on l. If P is any other point on l, then in the right angled triangle ADP, the hypotenuse $AP > AD$. We define the distance of a point P from a straight line l to be the perpendicular distance of P from l.

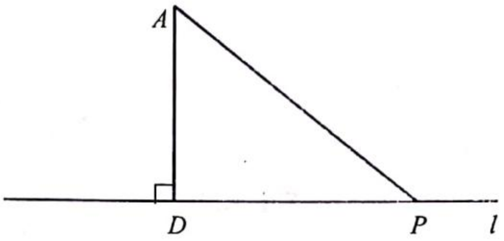


Fig. 3.29

Theorem 12. The locus of a point equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

Proof. Let A and B be two fixed points and P be a point such that $PA = PB$. Then one position of P is clearly the mid point D of the line segment AB. If P is any other position such that $PA = PB$, then comparing the triangles PAD and PBD (Fig. 3.30), we see that the triangles are congruent by the SSS theorem. As already seen, the median PD of the isosceles triangle PAB is perpendicular to AB. Conversely, it is clear that any point on the perpendicular bisector of AB is equidistant from A and B. Thus the locus of P which is equidistant from A and B is the perpendicular bisector of the line segment AB. ❑

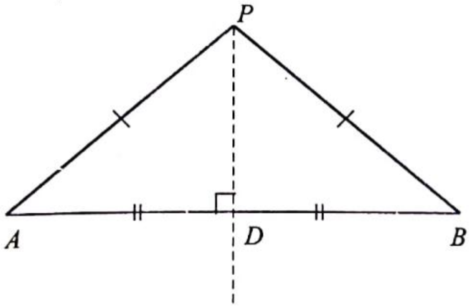


Fig. 3.30

Theorem 13. The locus of a point which is equidistant from two intersecting straight lines is the pair of bisectors of the angles formed by the two straight lines.

Proof. Let the straight lines AB and CD meet at O. Let l, m be the angular bisectors of the angles BOD and COB respectively. If P is any point on l, draw PM and PN perpendicular to AB and CD meeting them at M and N respectively (Fig. 3.31). The two right angled triangles OMP and ONP have the same hypotenuse OP and further $\angle POM = \angle PON$ since P lies on the bisector l. Therefore, $\Delta OMP \equiv \Delta ONP$, and so $PM = PN$. This means that P is equidistant from AB and CD. A similar argument shows that any point P on m is also equidistant from AB and CD. Also, if a point X is

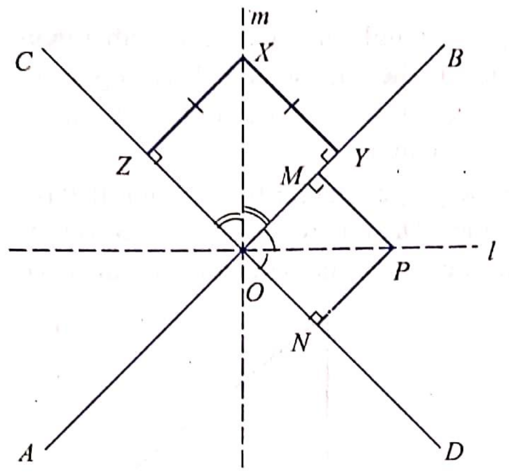


Fig. 3.31

equidistant from AB and CD, then $XY = XZ$ where Y and Z are the feet of the perpendiculars from X on AB, CD respectively. Again, comparing the right angled triangles XOZ and XOY, we note that hypotenuse OX is common and $XZ = XY$; therefore by RHS theorem $\Delta XOZ \equiv \Delta XOY$. Hence $\angle XOY = \angle XOZ$ or X lies on an angular bisector of the angles formed by AB and CD.

Thus the locus of a point equidistant from two intersecting lines is the pair of angular bisectors of the angles formed by the two straight lines. ❑

Remark. The angular bisectors l and m of the angles formed by AB and CD are mutually perpendicular (Fig. 3.32). If $\angle DOB = \alpha$, then $\angle BOC = 180° - \alpha$. Hence the angle between l and m is

$$\angle POB + \angle BOQ = \frac{\alpha}{2} + \frac{180° - \alpha}{2} = 90°$$

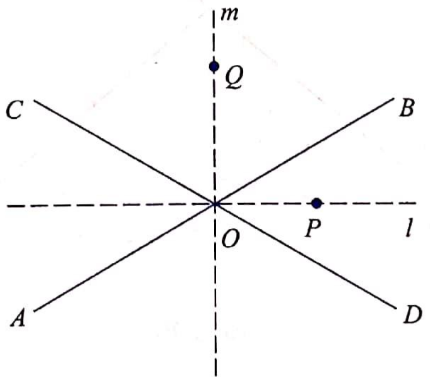


Fig. 3.32

EXERCISE 3.2

1. In a quadrilateral $ABCD$, the diagonals AD and BC meet at O. If it is given that $OA = OC$ and $OB = OD$, prove that $BC = AD$ and that $\angle ACB = \angle CAD$.
2. In ΔABC, D and E are the midpoints of the sides AB and AC respectively. If the perpendiculars at D and E to the sides AB and AC meet at O, prove that $OA = OB = OC$.

3. If two straight line segments AB and CD bisect each other at right angles, show that the sides of the quadrilateral are all equal.
4. Two isosceles triangles whose vertical angles are equal are placed so as to have their vertices coincident. (For an isosceles ΔABC with $AB = AC$, A is called the vertex of the ΔABC.) Prove that two of the lines joining their other angular points are equal.
5. ABC is a triangle and O is any point in it. Prove that
$$\angle BOC > \angle BAC.$$
6. If two isosceles triangles have equal bases and equal vertical angles prove that they are congruent.
7. AB and CD are two straight lines meeting at O and XY is another straight line. Show that in general two points can be found in XY which are equidistant from AB and CD. When is there only one such point?
8. $ABCD$ is a quadrilateral in which diagonals bisect each other. Show that B and D are equidistant from AC.
9. If in a quadrilateral $ABCD$, AC bisects angles A and C, show that AC is perpendicular to BD.
10. ABC and DBC are two triangles on the same base BC and on the same side of it such that $BA = CD$ and $BD = CA$. If AC and BD meet at O, prove that ΔOBA is congruent to ΔOCD.
11. Deduce from theorem 10, that the sum of any two sides is greater than the third side in any triangle.
12. In ΔABC, $AB > AC$. Let D on AB be such that $AD = AC$.
Then prove that
$$\angle ADC = (\angle B + \angle C)/2 \text{ and } \angle BCD = (\angle C - \angle B)/2.$$
13. The bisector of angle A of ΔABC meets BC at U.
Prove that if $AB > AC$ then $\angle AUC$ is acute.
14. With the same notations as in problem 13, prove that $AB > BU$ and $AC > CU$.
15. If the sides AB, BC, CD, DA of a quadrilateral $ABCD$ are in the descending order of magnitude, show that $\angle CDA > \angle CBA$.
16. If AD is the altitude through A of ΔABC, prove that $AB > AC$, $AB = AC$ or $AB < AC$ according as $BD > DC$, $BD = DC$ or $BD < DC$.
17. If X is any point on BC of ΔABC, prove that either AB or AC is greater than AX.
18. If BC is the greatest side of ΔABC, and D, E are points on BC, CA respectively prove that, $BC \geq DE$.
19. If BC is the greatest side of ΔABC, and E and F are points on AB, AC respectively prove that, $BC \geq EF$.
20. Prove that no straight line can be drawn within a triangle which is greater than the greatest side.
21. $ABCD$ is a quadrilateral having $AD = BC$. and $\angle ADC = \angle BCD$. If X is the midpoint of DC prove that $AX = BX$.
22. If the bisector of an angle of a triangle is perpendicular to the opposite side prove that the triangle is isosceles.
23. If two triangles are congruent prove that the straight lines joining the vertices to the midpoints of their bases are equal.
24. If two triangles are congruent prove that the perpendiculars from the vertex to the base of each are equal.
25. If triangles ABC and DEF are congruent and the bisectors of $\angle A$ and $\angle D$ meet BC and EF at X and Y respectively, then prove that $AX = DY$.

26. AU is the bisector of $\angle BAC$ and SUT is drawn perpendicular to AU meeting AB and AC at S and T respectively. Prove that ΔASU is congruent to ΔATU.

27. Through C the midpoint of a straight line segment AB, a straight line is drawn. Perpendiculars AD and BE are dropped upon it from A and B. Prove that $AD = BE$.

28. ABC is an isosceles triangle. The base BC is produced on either side to D and E so that $BD = CE$. Prove that $AD = AE$.

29. If the hypotenuse AC of a right angled ΔABC is of length $2AB$, prove that

$$\angle BAC = 2\angle ACB.$$

30. ABC is an isosceles triangle having $\angle B = \Delta C = 2\Delta A$. If BD bisecting $\angle B$ meets AC in D, prove that $AD = BC$.

3.3 PARALLEL STRAIGHT LINES

Definition 5. Two straight lines on a plane are *parallel* if they have either no common point or two common points; in the latter case the two straight lines coincide.

The fundamental property of parallel straight lines in Euclidean Geometry is the following

Axiom. For each point P and straight line l there is just one straight line through P parallel to it.

We observe that

1. any straight line l is parallel to itself;
2. if a straight line l_1 is parallel to a straight line l_2 then l_2 is parallel to l_1;
3. if a straight line l_1 is parallel to another straight line l_2 and if l_2 is parallel to l_3 then l_1 is parallel to l_3.

We write $l \parallel m$ to mean that l is parallel to m. A straight line drawn to cut two or more given straight lines is called a *transversal.* In the Fig. 3.33, the straight lines l_1 and l_2 are cut by the transversal m.

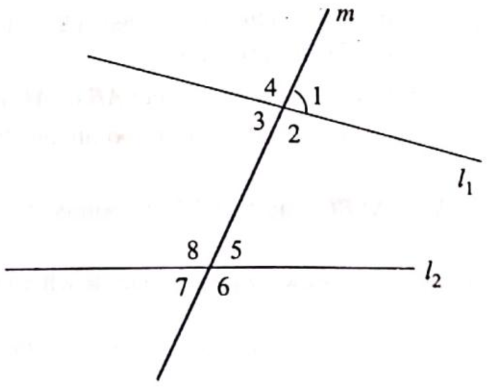


Fig. 3.33

In Fig. 3.33, the angles 2 and 8 are *alternate angles*; so are the angles 3 and 5. The angles 1 and 5 are *corresponding angles;* so are 4, 8; 2, 6; and 3,7.

Theorem 14. Suppose two straight lines are cut by a transversal and any one of the following three conditions hold, namely,

1. a pair of alternate angles are equal,
2. a pair of corresponding angles are equal,
3. a pair of interior angles on the same side of the transversal add upto 180°,

then the two straight lines are parallel.

Proof. The straight lines *AB* and *CD* are cut by the transversal *EF* meeting *AB* and *CD* at *G*, *H* respectively.

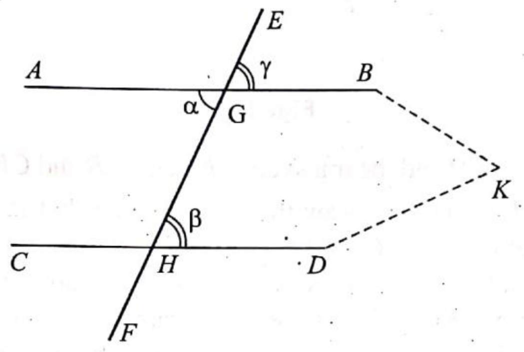


Fig. 3.34

1. Suppose the alternate angles $AGH = \alpha$ and $GHD = \beta$ are equal. If possible let *AB* and *CD* meet at a point *K*. If *K* is as shown in Fig. 3.34, then α is an exterior angle of ΔGKH for which β is an interior opposite angle. If *K* is on the other side of *GH*, then β will be an exterior angle of ΔGKH for which α is an interior opposite angle. Thus either $\alpha > \beta$ or $\beta > \alpha$, both contradicting our hypothesis that $\alpha = \beta$. Hence *AB* and *CD* do not meet and therefore are parallel.
2. Assume now that the corresponding angles $EGB = \gamma$ and $GHD = \beta$ are equal. In this case $\angle AGH = \angle EGB$ and hence $\alpha = \angle AGH = \angle EGB = \gamma = \angle GHD = \beta$. Now α and β is a pair of equal alternate angles and hence by (1) $AB \parallel CD$.
3. Suppose the sum of the interior angles *BGH* and *GHD* is 180°. Then $\alpha = \angle AGH = 180° - \angle BGH = \angle GHD$ and therefore again by (1) $AB \parallel CD$.

Corollary. If two straight lines *l* and *m* are both perpendicular to the same straight line *p*, then $l \parallel m$.

Proof. *l* and *m* are cut by the transversal *p* and the corresponding angles α and β are equal, each being equal to 90° (Fig. 3.35). Hence $l \parallel m$.

Theorem 15. If two parallel straight lines are cut by a transversal then

1. the alternate angles are equal
2. the corresponding angles are equal
3. the sum of the interior angles on the same side of the transversal is equal to 180°.

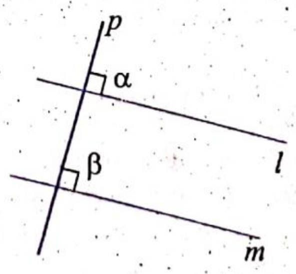


Fig. 3.35

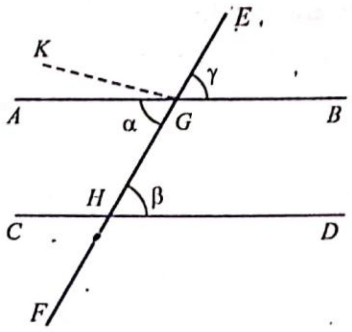


Fig. 3.36

Proof. In Fig. 3.36, $AB \parallel CD$ and the transversal EF cuts AB and CD at G, H respectively.

1. Suppose $\angle AGH \neq \angle GHD$. Draw the ray GK such that $\angle KGH = \angle GHD = \beta$. Then the straight lines KG and CD are cut by the transversal EF; and by construction the alternate angles KGH and GHD are equal. Therefore by Theorem 14 $KG \parallel CD$. Now KG and AB are two straight lines parallel to CD passing through G contradicting our axiom on parallel straight lines. Hence our supposition that $\angle AGH \neq \angle GHD$ is false and we must have the alternate angles AGH and GHD as equal angles.
2. The equality of the alternate angles AGH and GHD implies that the corresponding angles EGB and GHD are equal.
3. Again the equality of the alternate angles AGH and GHD implies that the sum of the interior angles BGH and GHD is 180°. ❑

Corollary. If a straight line is perpendicular to one of two or more parallel straight lines, then it is also perpendicular to the other parallel straight lines.

Proof. Suppose l, m, n are parallel straight lines and p is a straight line perpendicular to l. For the transversal p cutting the parallel lines l and m, the equality of the alternate angles α and β implies that $90° = \alpha = \beta$ (Fig. 3.37). Hence $p \perp m$. The same reasoning tells us that p is perpendicular to all the straight lines parallel to l. ❑

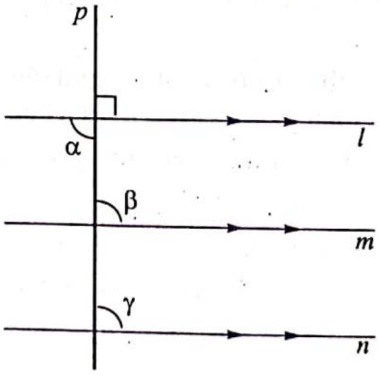


Fig. 3.37

Theorem 16. The sum of the interior angles of a triangle is 180°.

Proof. Let ABC be a triangle. Produce BC to D and draw $CE \parallel BA$ (Fig. 3.38). Then $\angle ABC = \angle ECD$ being corresponding angles for the transversal BD cutting the parallel lines AB and EC. Again the angles BAC and ACE are alternate angles and therefore they are equal. Thus we have

$$\angle A + \angle B + \angle C = \angle ACE + \angle ECD + \angle ACB = 180°.$$ ❑

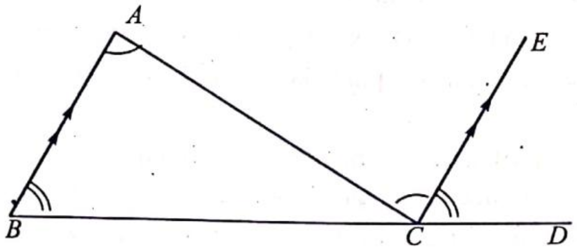


Fig. 3.38

Corollary. An exterior angle of a triangle is equal to the sum of the interior opposite angles.

Proof. For ΔABC, we note that the exterior $\angle DCA = \angle DCE + \angle ECA$ (Fig. 3.38) $= \angle A + \angle B$ (see the proof of Theorem 16). Also $\angle DCA = 180° - \angle BCA$

$= 180° - \angle C = \angle A + \angle B$ (since $\angle A + \angle B + \angle C = 180°$). ❑

Theorem 17. The sum of the interior angles of any convex polygon having n sides is $(2n - 4)$ right angles. [Recall that two right angles = 180° and that a polygon is a non-self intersecting closed polygonal line $A_1A_2A_3 \ldots A_nA_1$ consisting of line segments A_1A_2, A_2A_3,.... $A_{n-1}A_n$ and A_nA_1; a *convex polygon* is a polygon for which the line segment joining any two points in its interior lies entirely within the polygonal region. Note that a polygon is convex iff it entirely lies on one side of any straight line containing a side of the polygon.]

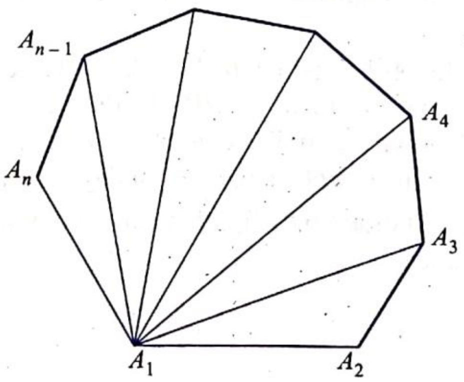


Fig. 3.39

Proof. If $A_1A_2 \ldots A_nA_1$ is any convex polygon, Join A_1 with its other vertices to get $(n - 2)$ triangles namely,

$$\Delta A_1A_2A_3,\ \Delta A_1A_3A_4,\ \Delta A_1A_4A_5,\ \ldots \Delta A_1A_{n-1}A_n.$$

We note that the sum of the interior angles of the polygon, the sum of the angles of all the $(n - 2)$ triangles $= 2(n - 2)$ right angles. ❑

Corollary. The sum of the exterior angles of a convex polygon is 360° or four right angles.

Proof. The sum of all the exterior angles and the sum of all the interior angles of a convex polygon of n sides is $(180n)°$ since each exterior angle is supplementary to its corresponding interior angle. Therefore the sum of the exterior angles $= n180° - (n - 2)180° = 2(180°) = 360°$. ❑

Note (*i*) The sum of the interior angles of a convex quadrilateral (convex polygon of four sides) is $2(4) - 4 = 4$ right angles $= 360°$.

(*ii*) The theorem is true for non-convex polygon also (see Exercise 3.3, problem...).

Definition 6. A *parallelogram* is a quadrilateral with both pairs of opposite sides being parallel.

Theorem 18. In a parallelogram, opposite sides are equal, opposite angles are equal, the sum of any pair of adjacent angles is 180° and the diagonals bisect each other.

Proof. Let $ABCD$ be a parallelogram. Then $AB \parallel DC$ and $AD \parallel BC$. The triangles ABC and CDA are congruent since AC is a common side, $\angle CAB = \angle ACD$ (alt angles), $\angle ACB = \angle CAD$ (alt angles) (by the ASA theorem). Therefore $AB = CD$ and $AD = BC$. Thus the opposite sides are equal.

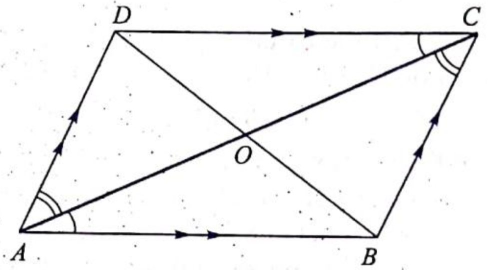


Fig. 3.40

Also $\Delta ABC \equiv \Delta CDA$ implies that $\angle ABC = \angle CDA$, *i.e.*, the opposite angles B and D are equal. Similarly, the congruence of the trianlges BAD and DCB implies that $\angle A = \angle C$.

Comparing the triangles AOD and COB (Fig. 3.40) we see that $\angle AOD = \angle COB$ (vertically opposite angles), $\angle DAO = \angle BCO$ (alternate angles) and $AD = BC$. Therefore, $\Delta AOD \equiv \Delta COB$ which implies that $AO = OC$. Similarly, one can show that $BO = OD$. Thus the diagonals bisect each other, in any parallelogram. ❑

Corollary. If two straight lines are parallel, then any two points on one line are at the same distance from the other.

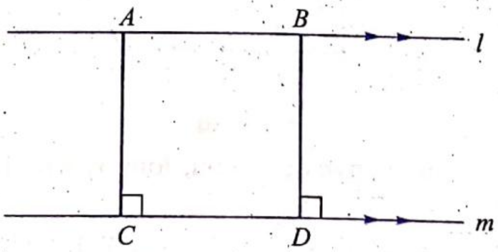


Fig. 3.41

Proof. Let l and m be any two parallel straight lines. If A and B are any two points on l, draw the perpendiculars AC and BD to m from A and B respectively (Fig. 3.41). It is required to prove that $AC = BD$. We note that $ABDC$ is a parallelogram and hence we have $AC = BD$. ❑

Theorem 19. In a convex quadrilateral, the following are equivalent:

1. the quadrilateral is a parallelogram

2. the opposite sides are equal
3. the opposite angles are equal
4. the diagonals bisect each other.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 18.

To prove (2) $\Rightarrow$ (3):

We assume that $AB = CD$ and $AD = BC$. We want to prove that $\angle A = \angle C$ and $\angle B = \angle D$. In ΔABC and ΔCDA we have $AB = CD$, $BC = AD$ and AC is a common side. Therefor by the SSS theorem $\Delta ABC \equiv \Delta CDA$ and so $\angle B = \angle D$. Similarly one can show that $\angle A = \angle C$. Thus (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (4):

We assume that $\angle A = \angle C$ and $\angle B = \angle D$ and we want to prove that the diagonals AC and BD bisect each other. Suppose $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$. Then we have $2\alpha + 2\beta = 2(\alpha + \beta) = 360°$ and hence $\alpha + \beta = 180°$. The straight lines AB and DC are cut by the transversal AD; the sum of the interior angles $\angle CDA + \angle DAB = \beta + \alpha = 180°$ and hence $AB \parallel DC$. Similarly $AD \parallel BC$. Therefore, the alternate angles OCD and OAB are equal; also $\angle ODC = \angle OBA$. Further $AB \parallel DC$ and $AD \parallel BC$ implies that $ABCD$ is a parallelogram and hence $AB = CD$. This means that $\Delta OAB \equiv \Delta OCD$ and hence $AO = OC$ and $BO = OD$. This proves that (3) $\Rightarrow$ (4).

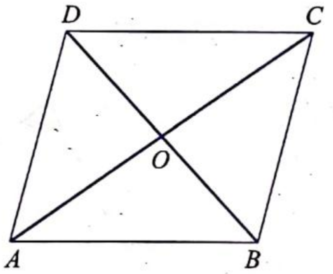


Fig. 3.42

To prove (4) $\Rightarrow$ (1):

We assume that the diagonals AC and BD bisect each other. We want to prove that $ABCD$ is a parallelogram. In triangles AOB and COD we have $\angle AOB = \angle COD$ (vert. opp. angles), $AO = OC$, $BO = OD$. Therefore, by the SAS test $\Delta AOB \equiv \Delta COD$. This implies that the alternate angles OCD and OAB are equal. Therefore, $AB \parallel CD$. Similar argument shows that $AD \parallel BC$ and thus $ABCD$ is a parallelogram. ❑

Theorem 20. If there are three or more parallel straight lines, and the intercepts made by them on a transversal are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

Proof. Suppose the three parallel straight lines l, m, n are cut by the transversals ACE and BDF and $AC = CE$. We want to prove that $BD = DF$ (Fig. 3.43). Draw AG and CH parallel to BF meeting m at G and n at H respectively. Then in triangles CAG and ECH we have $\angle CAG = \angle ECH$ (cor. angles), $\angle ACG = \angle CEH$ (cor. angles) and $AC = CE$. Therefore by the ASA theorem $\Delta CAG \equiv \Delta ECH$; and so $AG = CH$. But by construction $ABDG$ and $CDFH$ are parallelograms. This means that $AG = BD$, $CH = DF$ and hence $BD = DF$.

Theorem 21. In any triangle ABC, the line FE joining the mid points of AB and AC is parallel to BC and $FE = (1/2)\ BC$.

Proof. Draw $AX \parallel BC$ (Fig. 3.44). Then if FY is the straight line parallel to BC, the transversal AB has equal intercepts AF and FB on the parallel lines AX, FY and BC. By Theorem 20, AC also should make equal intercepts on these parallel straight lines and hence E lies on FY. This means that $FE \parallel BC$.

Draw $CY \parallel BA$ meeting FE at Y (Fig. 3.45). Then $BCYF$ is a parallelogram and $FY = BC$. Further ΔAFE and ΔCYE are congruent since $\angle AEF = \angle CEY$ (vert. opp. angles), $\angle FAE = \angle YCE$ (alt. angles) and $AE = EC$.

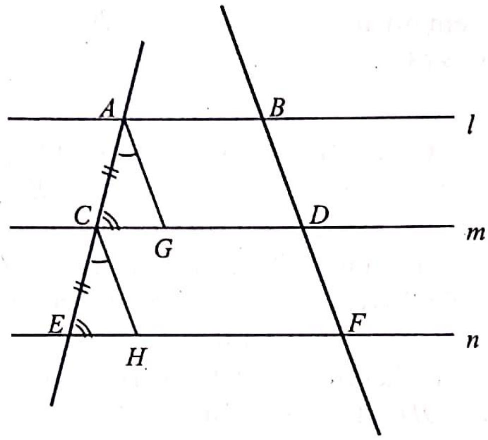


Fig. 3.43

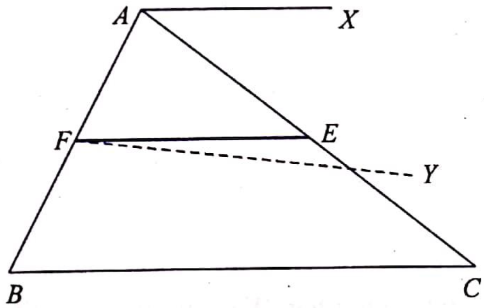


Fig. 3.44

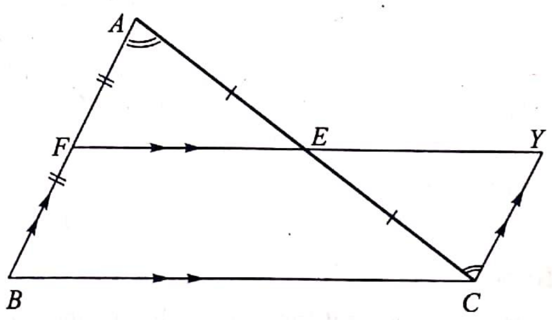


Fig. 3.45

Therefore we have $FE = EY$ and $2FE = FY = BC$. Thus $FE = (1/2)$ BC. ❑

Theorem 22. If in ΔABC, a straight line is drawn parallel to BC through the midpoint F of AB, then it bisects the side AC.

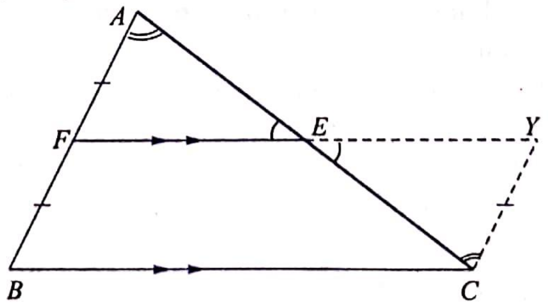


Fig. 3.46

Proof. Let the straight line through F parallel to BC meet AC at E. It is required to prove that $AE = EC$. Draw $CY \parallel BA$ meeting FE produced at Y. We have $\angle AEF = \angle CEY$ (vert. opp. angles), $\angle FAE = \angle YCF$ (alt. angles) and $AF = FB = CY$ (since $BCYF$ is a parallelogram). Therefore

$$\Delta AFE \equiv \Delta CYE \text{ and this gives } AE = EC.$$ ❑

EXERCISE 3.3

1. In a parallelogram prove that
   (*i*) if one angle is a right angle then all the angles are right angles.
   (*ii*) opposite angles are equal
2. If each angle of a rectilinear figure is $7/8\pi$ radians, find the number of sides.
3. Prove that the bisectors of angles in a parallelogram form a rectangle.
4. In ΔABC the internal bisectors of angles B and C meet at I. Prove that $\angle BIC = 90° + \angle A/2$.
5. In problem 4, if the external bisectors of angles B and C meet at X, then $\angle BXC = 90° - \angle A/2$.
6. In ΔABC perpendiculars AD and BF are drawn to BC and CA respectively to meet at the point H. Prove that $\angle AHE = \angle C$.
7. $ABCD$ is a trapezium with side $BC \parallel AD$. If E is the midpoint of AB and the line through E parallel to DC meets AD and BC at X and Y respectively, prove that $ABCD$ and $XYCD$ have equal areas.
8. If two quadrilaterals $ABCD$ and $PQRS$ have angles A, B, C, D equal to angles P, Q, R, S respectively and $AB = EF$, $DC = HG$ and if AD is not parallel to BC, prove that the quadrilaterals are congruent.
9. Equilateral triangles BAD and CAE are drawn on the sides AB, AC of an equilateral ΔABC externally to the triangle. Show that D, A, E are collinear.
10. If on the sides BC, CA, AB of an equilateral triangle, the equilateral triangles BCX, CAY and ABZ are drawn externally to ΔABC prove that XYZ is also equilateral.
11. Equilateral triangles ABX and ACY are described on sides AB, AC of a ΔABC externally to ΔABC. Prove that $CX = BY$.
12. If in a parallelogram $ABCD$, the diagonal AC bisects $\angle A$, then prove that $ABCD$ is a rhombus.
13. Show how to find points D and E on the side AB, AC of ΔABC such that $DE \parallel BC$ and $DE = BD$.
14. Let $ABCDE$ be a regular pentagon. If the internal angular bisectors of angles A and B meet at O, prove that OC, OD, OE also bisect angles C, D and E.
15. If in ΔABC BC is the greatest side, prove that $\angle A > \angle 60°$.
16. Let D be the midpoint of the side BC of ΔABC. Prove that if $AD > BD$ then $\angle A$ is acute; else if $AD < BD$, then $\angle A$ is obtuse.
17. Let D be the midpoint of the hypotenuse BC of the right angled ΔABC. Prove that $2AD = BC$.
18. If the sum of the distances of any vertex of a quadrilateral from the other three is same for all the four vertices, prove that the quadrilateral is a rectangle.
19. $ABCD$ is a parallelogram and O is any point. The parallelograms $OAEB$, $OBFC$, $OCGD$, $ODHA$ are completed. Show that $EFGH$ is a parallelogram.
20. Any point X is taken on the side BC of ΔABC. Prove that AX is bisected by the straight line joining the midpoints of AB and AC.

21. Suppose the straight line AB of ΔABC is bisected at C and the perpendiculars AX, BY, CZ are drawn to any straight line OP. Prove that

(*i*) if A, B are on the same side of OP, then $2CZ = AX + BY$.

(*ii*) if A, B are on the opposite sides of OP, then $2CZ$ = difference of AX and BY.

22. Prove that the straight lines joining the midpoints of the diagonals of a trapezium is parallel to the parallel sides.

23. Prove that in any quadrilateral, the midpoints of the sides form the vertices of a parallelogram.

24. Prove that the lines joining the midpoints of opposite sides of a quadrilateral and the line joining the midpoints of diagonals are concurrent.

25. Let X be the midpoint of the side AB of ΔABC. Let Y be the midpoint of CX. Let BY cut AC at Z. Prove that $AZ = 2ZC$.

26. A is a given point and P is any point on a given straight line. If $AQ = AP$ and AQ makes a constant angle with AP, find the locus of Q.

27. ABC is a equilateral triangle with vertex A fixed and B moving in a given straight line. Find the locus of C.

28. If a trisector of an exterior angle of a triangle is parallel to a bisector of an interior angle, prove that the other trisector of the exterior angle is parallel to a trisector of an interior angle.

3.4 SOME PROPERTIES OF A TRIANGLE

Theorem 23. The perpendicular bisectors of the three sides of a triangle concur at a point.

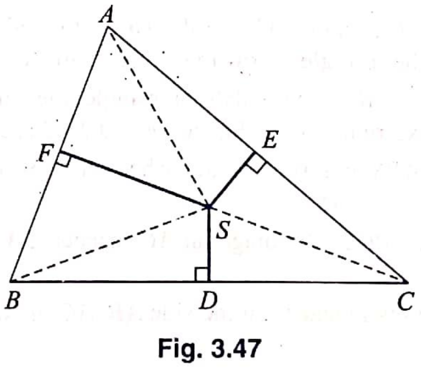


Fig. 3.47

Proof. Let ABC be a triangle; let D, E, F be the midpoints of the sides BC, CA, AB respectively. Suppose the $\perp$ bisectors of BC and CA meet at S. Then it is required to prove that $SF \perp AB$. S being a point on the perpendicular bisector of BC, we have $SB = SC$ (Theorem 12). Again, S lies on the perpendicular bisector of CA implies that $SC = SA$. Thus $SA = SB = SC$. In triangles ASF and BSF we have $AF = BF$ (Hypothesis), $SA = SB$ (proved above) and $SF = SF$ (common side). Therefore, $\Delta ASF \equiv \Delta BSF$. This implies that $\angle AFS = \angle BFS = 90°$ or $SF \perp AB$. ❑

Note. The point of concurrence S, of the perpendicular bisectors of the sides of a triangle ABC is known as the *circumcentre* of ΔABC.

Theorem 24. The bisectors of the three angles of a triangle meet at a point.

Proof. Let ABC be a triangle and let the bisectors of the angles B and C meet at I (Fig. 3.48). It is required to prove that AI bisects $\angle A$. Draw IX, IY, IZ perpendicular to

BC, CA, AB respectively (Fig. 3.48). Since I lies on the bisector BI of $\angle B$ we have $IX = IZ$ (Theorem 13). Similarly, the fact that I lies on the bisector of $\angle C$ implies that $IX = IY$. Thus $IX = IY = IZ$. Now $IY = IZ$ implies, again by Theorem 13, that I lies on the bisector of $\angle A$. Hence the bisectors of the three angles of a triangle concur at a point. ❑

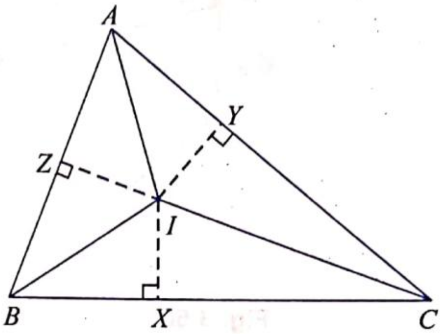


Fig. 3.48

Note. The point of concurrence I, of the internal bisectors of the angles of a triangle is known as the *incentre* of the triangle.

Theorem 25. The three medians of a triangle meet at a point and the point of concurrence trisects each median.

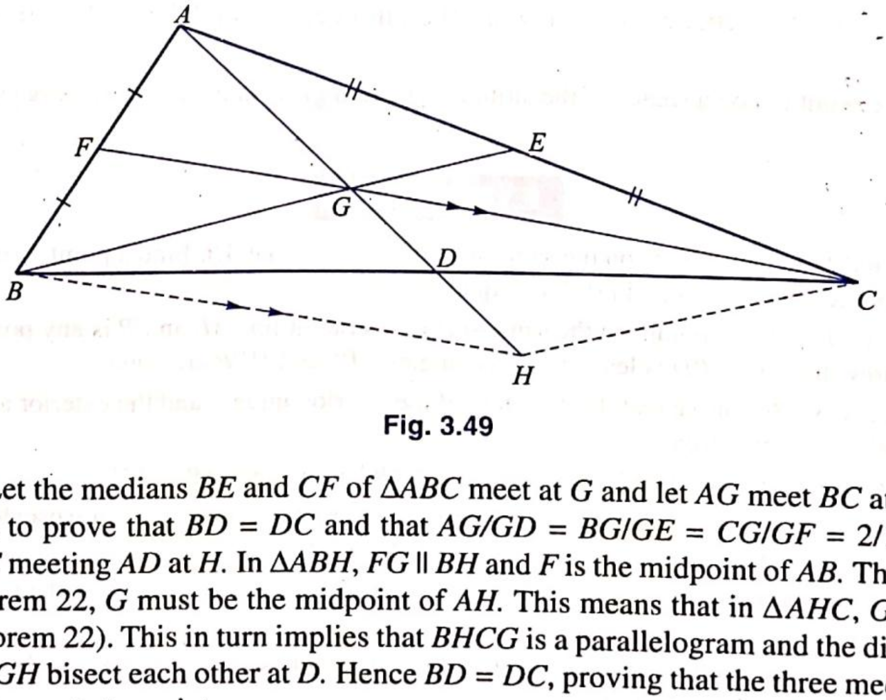


Fig. 3.49

Proof. Let the medians BE and CF of ΔABC meet at G and let AG meet BC at D. It is required to prove that $BD = DC$ and that $AG/GD = BG/GE = CG/GF = 2/1$. Draw $BH \parallel FC$ meeting AD at H. In ΔABH, $FG \parallel BH$ and F is the midpoint of AB. Therefore, by Theorem 22, G must be the midpoint of AH. This means that in ΔAHC, $GE \parallel HC$ (by Theorem 22). This in turn implies that $BHCG$ is a parallelogram and the diagonals BC and GH bisect each other at D. Hence $BD = DC$, proving that the three medians of a triangle meet at a point.

Further, as already observed, $AG = GH$ and $GD = DH = (1/2)\ GH$ implies that $AG/GD = 2/1$. Similarly BE and CF are also trisected by G. ❑

Note. G is known as the *centroid* of ΔABC.

Theorem 26. The altitudes of a triangle meet at a point.

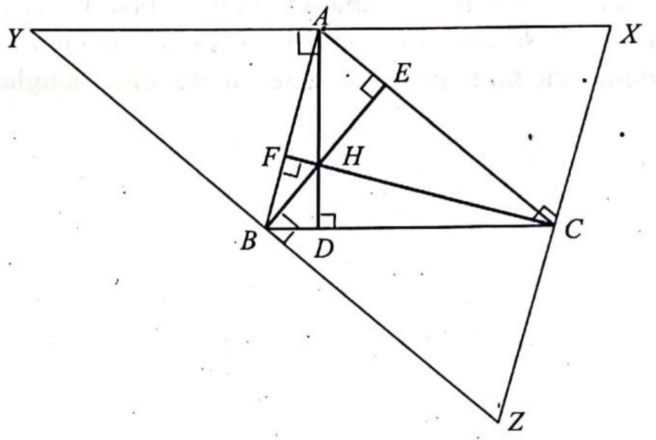


Fig. 3.50

Proof. Let ABC be a triangle. Draw XY, YZ, ZX through A, B, C parallel to BC, CA, AB respectively. Then $BCAY$ and $BCXA$ are parallelograms and hence $YA = BC = AX$ or A is the midpoint of XY. Similarly, B is the midpoint of YZ and C is the midpoint of ZX. If the perpendiculars AD, BE, CF are drawn to the sides XY, YZ, ZX respectively (Fig. 3.50), then by Theorem 23, AD, BE and CF meet at the circumcentre H of ΔXYZ. Now $XY \parallel BC$, $YZ \parallel CA$ and $ZX \parallel AB$ implies that AD, BE, CF are also perpendicular to BC, CA, AB of ΔABC. Hence, they are the altitudes of ΔABC and they meet at the point H.

Note. The point of concurrence of the altitudes of a triangle is known as the *orthocentre* of the triangle.

EXERCISE 3.4

1. C and D are two points on the same side of a straight line AB. Find a point X on AB such that the angles CXA and DXB are equal.
2. C and D are two points on the same side of a straight line AB and P is any point on AB. Show that $PC + PD$ is least when the angles CPA and DPB are equal.
3. In any ΔABC, prove that the bisectors of the interior angle A and the exterior angles at B and C are concurrent.
4. If the medians BE and CF are equal in a ΔABC prove that $AB = AC$.
5. If P is any point on a straight line drawn through the vertex A of an isosceles ΔABC, parallel to the base, prove that $PB + PC > AB + AC$.
6. If S is the circumcentre of ΔABC prove that $\angle BSD = \angle BAC$ where D is the midpoint of BC.
7. If $ABCD$ is a parallelogram, prove that the circumcentres of the triangles ABC and ADC are at the same distance from AC.
8. X is any point in the base BC of an isosceles ΔABC; P and Q are the circumcentres of the triangles ABX and ACX. Prove that $APXQ$ is a parallelogram.
9. Let D, E, F be the feet of the altitude from A, B, C in a ΔABC. Prove that the perpendicular bisector of EF also bisects BC.
10. Suppose the diagonals of a quadrilateral $ABCD$ meet at a point O; then prove that the circumcentres of the four triangles OAB, OBC, OCD and ODA form a parallelogram.

11. If H is the orthocentre of ΔABC, prove that A is the orthocentre of ΔBHC.

12. Prove that the circumcentre S of ΔABC where A', B', C' where A', B', C' are the midpoint of BC, CA, AB respectively.

13. If S is the circumcentre of a ΔABC and D, E, F are the feet of the altitudes of ΔABC then prove that $SB \perp DF$.

14. Let P be any point inside a regular polygon. If d_i is the distance of P from the ith side of the polygon, prove that $d_1 + d_2 \ldots + d_n$ = constant, where n is the number of sides of the polygon.

15. If I is the incentre and S is the circumcentre of ΔABC prove that $\angle IAS$ half the difference in $\angle B$ and $\angle C$.

16. AB and CD are two fixed straight lines and a variable straight line cuts them at X and Y respectively. The angular bisectors of $\angle AXY$ and $\angle CXY$ meet at P. Find the locus of P.

17. I is the incentre of ΔABC. X and Y are the feet of the perpendiculars from A to BI and CI. Prove that XY is parallel to BC.

3.5 SIMILAR TRIANGLES

Theorem 27. Parallelograms on the same base and between the same parallels are equal in area.

Proof. Let $ABCD$ and $ABXY$ be two parallelograms having the same base AB and lying between the same parallels, namely the straight line AB and the straight line YC. (Fig. 3.51) The parallelogram $ABCD$ = Trapezium $ABCY - \Delta ADY$ and the parallelogram $ABXY$ = Trapezium $ABCY - \Delta BCX$ (in area). Therefore to prove that the two parallelograms $ABCD$ and $ABXY$ have equal areas, it is enough to prove that ΔADY and ΔBCX have equal areas. We have $\angle ADY = \angle BCX$, (cor. angles) $\angle AYD = \angle BXC$ (cor. angles) and $AY = BX$ (since $ABXY$ is a parallelogram). Therefore by the ASA theorem the two triangles ADY and BCX are congruent. So they have equal areas. ❑

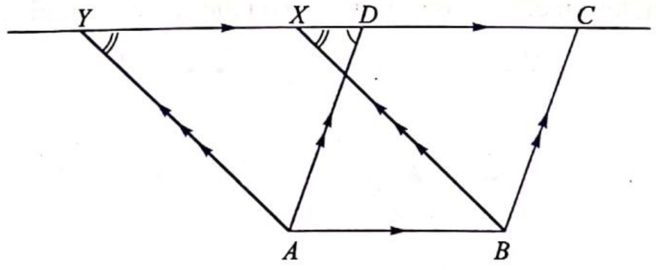


Fig. 3.51

Note. The above proof works even when Y lies between C and D. (Fig. 3.52)

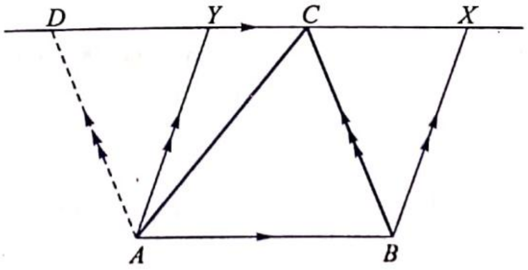


Fig. 3.52

Corollary. The area of a parallelogram is equal to the area of the rectangle whose adjacent sides are equal to the base and the altitude of the parallelogram. In other words, the area of a parallelogram is the product of its base and the altitude.

Proof. Let $ABCD$ be a parallelogram and $ABXY$ be the rectangle on the base AB and with BX equal to the altitude of the parallelogram (Fig. 3.53). Then the parallelograms $ABCD$ and $ABXY$ have the same base and lie between the same parallels. Therefore by Theorem 27, the rectangle $ABXY$ and the parallelogram $ABCD$ have the same area. ❑

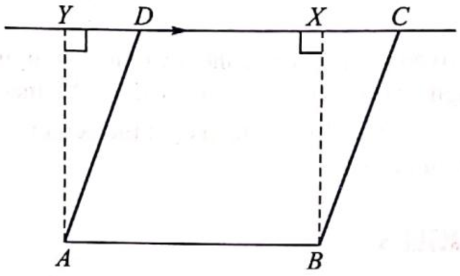


Fig. 3.53

Theorem 28. If a parallelogram and a triangle are on the same base and lie between the same parallels, then the area of the triangle is half that of the parallelogram.

Proof. Consider the ΔABC and the parallelogram $ABXY$ having the same base AB and lying between the same parallels AB and YX. Draw $AD \parallel BC$ (Fig. 3.54) so that $ABCD$ is a parallelogram. Then by Theorem 27, the two parallelograms $ABCD$ and $ABXY$ have equal areas. Now, the triangles ABC and CDA are congruent (Why ?). Hence, area of ΔABC = half the area of parallelogram $ABCD$ = half the area of parallelogram $ABXY$. ❑

Corollary. Triangles on equal bases and between the same parallels are equal in area.

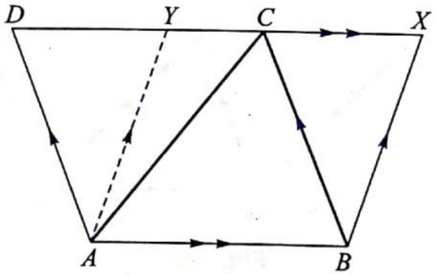


Fig. 3.54

Proof. Exercise. ❑

Note. (1) By the corollary to Theorem 27, the area of a parallelogram $ABCD$

$= AB \times$ altitude through A

$=$ base $\times$ height.

(2) Area of ΔABC

$$= \frac{1}{2} \text{ area of the parallelogram } ABCD$$

$$= \frac{1}{2} (\text{base} \times \text{height}) \text{ (Fig. 3.55).}$$

(3) The ratio of the areas of two triangles of equal altitudes is equal to the ratio of their bases.

(4) The ratio of the areas of two triangles on equal bases is equal to the ratio of their altitudes.

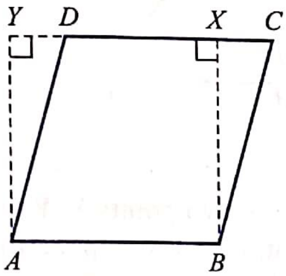


Fig. 3.55

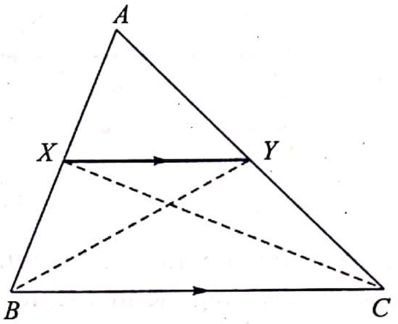


Fig. 3.56

Theorem 29. If a straight line is drawn parallel to one side of a triangle, then it divides the other two sides proportionally. Also, conversely, if a straight line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof. Let XY be a straight line parallel to BC meeting AB, AC at X, Y. (Fig. 3.56). We want to prove that $AX/XB = AY/YC$. The triangles AXY and BXY have equal altitudes and hence

$$\frac{\Delta AXY}{\Delta BXY} = \frac{AX}{XB}$$

Similarly
$$\frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}$$

Now the triangles BXY and CXY have the same base XY and are between the same parallels. Hence $\Delta BXY = \Delta CXY$.

This gives
$$\frac{AX}{XB} = \frac{\Delta AXY}{\Delta BXY} = \frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}.$$

Conversely, suppose the straight line XY meets AB, AC at X, Y and

$$\frac{AX}{XB} = \frac{AY}{YC}.$$

We want to prove that $XY \parallel BC$. Again, we have

$$\frac{\Delta AXY}{\Delta BXY} = \frac{AX}{XB} \quad \text{and} \quad \frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}$$

By our hypothesis, $\frac{AX}{XB} = \frac{AY}{YC}$ and therefore $\Delta BXY = \Delta CXY$. Now BXY and CXY are two triangles of equal areas on XY and are on the same side of XY. Hence $XY \parallel BC$. ❑

Note. The points X, Y in the above theorem may lie on AB, AC produced as in Fig. 3.57. Still the same proof works.

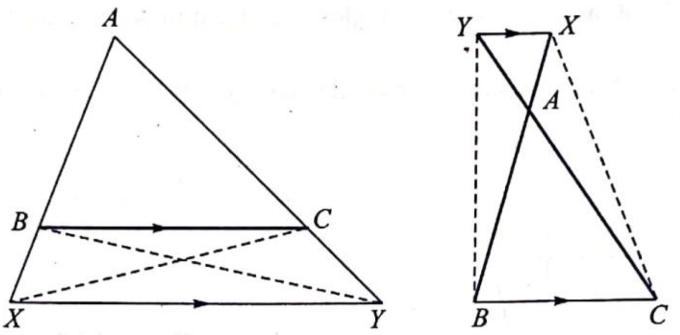


Fig. 3.57

Theorem 30. Given any ratio $\lambda/1 > 0$ there exist exactly two points X, Y on a given straight line AB dividing the line segment AB in the ratio $\lambda : 1$, one point dividing it internally and the other externally. However if $\lambda = 1$, there is no point on the straight line AB which divides the segment externally in the ratio $\lambda : 1$.

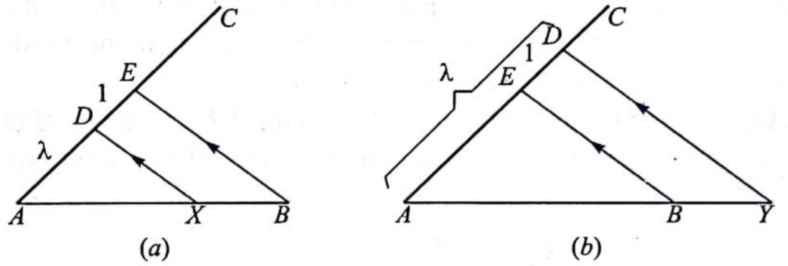


Fig. 3.58

Proof. First, let us prove that there cannot be two points X, X' on AB dividing the segment AB internally in the same ratio $\lambda : 1$. Suppose X, X' are two such points then

$$\frac{AX}{XB} = \frac{AX'}{X'B} = \lambda.$$

This implies that $$\frac{AX + XB}{XB} = \frac{AX' + X'B}{X'B}.$$

This gives $$\frac{AB}{XB} = \frac{AB}{A'B}$$ and hence $XB = X'B$ or $X = X'$.

Similarly one observes that there cannot be two points Y, Y' on AB dividing AB externally in the ratio $\lambda : 1$. Now we prove that there exist points X, Y on AB dividing the line segment AB internally and externally in the ratio $\lambda : 1$. Take a convenient ray AC as shown in Fig. 3.58. Cut off points D, E on AC such that $AD = \lambda$ and $DE = 1$, for internal division as shown in Fig. 3.58(*a*); and such that $AD = \lambda$, $ED = 1$ for external division as in Fig. 3.58(*b*). Draw $DX \parallel EB$ meeting AB at X for internal division; and draw $DY \parallel EB$ meeting AB produced at Y for external division. Then by Theorem 29,

we have $$\frac{AX}{XD} = \frac{AD}{DE} = \frac{\lambda}{1} \quad \text{and} \quad \frac{AY}{YB} = \frac{AD}{DE} = \frac{\lambda}{1}$$

Hence the theorem. If $\lambda = 1$, then there exists no point Y on the straight line AB produced such that $AY/YB = 1$; for, any such point Y satisfies either $AY > YB$ or $AY < YB$ (Fig. 3.59). ❑

Fig. 3.59

Note. Sometimes it is convenient to attach signs to lengths of directed line segments. Taking the direction A to B to be positive, we note that length $BA = -$ length AB. Therefore if P divides AB internally then $\frac{AP}{PB} > 0$ as AP and PB have the same direction. On the other hand if P divides AB externally, then $\frac{AP}{PB} < 0$ since AP and PB have opposite directions.

Theorem 31. The internal (or external) bisector of an angle of a triangle divides the opposite side internally (or externally) in the ratio of the sides containing the angle.

Proof. Let AD be the internal (external) bisector of $\angle A$ of ΔABC meeting BC(BC produced) at D. It is required to prove that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

Draw $CE \parallel AD$ meeting BA or BA produced at E.

Let F be a point on BA or BA produced (See Fig 3.60(a) and Fig. 3.60(b)).

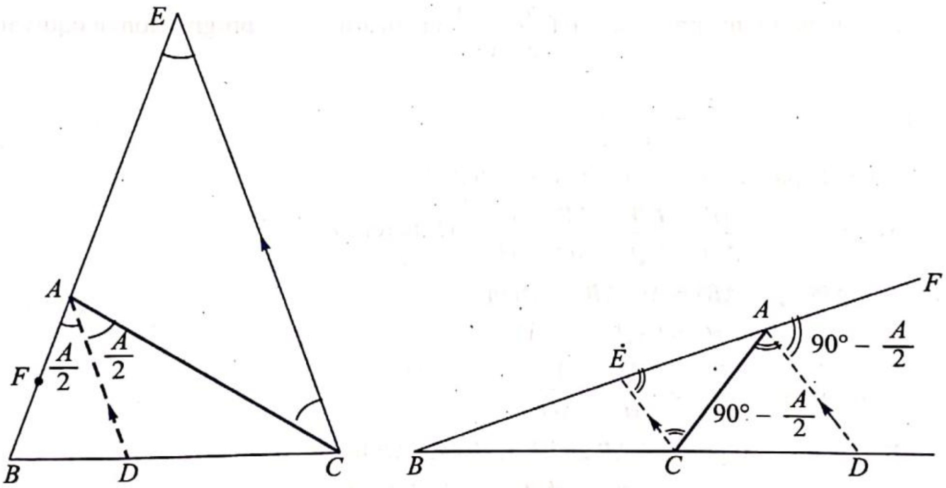


Fig. 3.60

Then $$\angle FAD = \angle AEC = \frac{\angle A}{2} \text{ or } \left(90° - \frac{\angle A}{2}\right)$$

Also $\angle DAC = \angle ACE = \frac{\angle A}{2}$ or $\left(90° - \frac{\angle A}{2}\right)$. Thus in $\angle AEC$, we have $\angle AEC = \angle ACE$ and hence $AE = AC$. In ΔBCE, we have $AD \parallel EC$ and therefore by Theorem 29, we get

$$\frac{BD}{DC} = \frac{BA}{AE} = \frac{BA}{AC} \text{ (Since } AE = AC \text{ as already observed)}$$

Thus $$\frac{BD}{DC} = \frac{AB}{AC}$$ ❑

Definition 7. If a line segment AB is divided internally and externally in the same ratio at P and Q respectively, then AB is said to be divided harmonically at P and Q. P and Q are called *harmonic conjugates* with respect to AB.

Fig. 3.61

In Fig. 3.61, P and Q are harmonic conjugates with respect to AB.

$$\frac{AP}{PB} = \frac{AQ}{BQ}$$

Note. 1. If P and Q are harmonic conjugates with respect to AB, then A and B are harmonic conjugates with respect to PQ.

For, $\quad \dfrac{AP}{PB} = -\dfrac{AQ}{QB}$

implies that $\quad \dfrac{PA}{AQ} = -\dfrac{AP}{AQ} = \dfrac{PB}{QB} = -\dfrac{PB}{BQ}$.

2. If P and Q divide AB harmonically, then AP, AB and AQ are in harmonic progression. (Three numbers a, b, c are in arithmetic progression if $2b = a + c$. Three numbers x, y, z are in harmonic progression if $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in arithmetic progression or equivalently

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

(See chapter 15 for more on progressions).

We have $\quad \dfrac{AP}{AQ} = \dfrac{PB}{BQ} = \dfrac{AB - AP}{AQ - AB}$ and therefore,

$$AP(AQ - AB) = AQ(AB - AP) \text{ or}$$

$$2AP \,.\, AQ = (AQ + AP)AB$$

This gives $\quad \dfrac{2}{AB} = \dfrac{1}{AP} + \dfrac{1}{AQ}$.

3. If O is the mid point of AB and P, Q divide AB harmonically then $OB^2 = OP.\, OQ$.

We have $\quad \dfrac{AP}{PB} = \dfrac{AQ}{BQ} \quad$ or $\quad \dfrac{AP + PB}{AP - PB} = \dfrac{AQ + BQ}{AQ - BQ}$

Therefore, $\quad \dfrac{AB}{(AO + OP) - (OB - OP)} = \dfrac{(AO + OQ) + (OQ - OB)}{(AO + OQ) - (OQ - OB)}$

Simplifying, $\quad \dfrac{AB}{2OP} = \dfrac{2OB}{2OP} = \dfrac{2OQ}{2OB} \quad$ or $\quad OB^2 = OP \,.\, OQ$.

Definition 8. Two triangles are *similar* if the three angles of the one are equal to the three angles of the other taken in order and the sides about the equal angles are proportional.

In other words $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ are similar if $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}.$$

We write $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_2B_2C_2$ to mean that the two triangles are similar. We observe the following.

1. Any triangle is similar to itself.
2. If $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_2B_2C_2$ then $\Delta A_2B_2C_2 \mathrel{|||} \Delta A_1B_1C_1$.
3. If $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_2B_2C_2$ and $\Delta A_2B_2C_2 \mathrel{|||} \Delta A_3B_3C_3$ then $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_3B_3C_3$.
4. If two triangles are congruent, then they are also similar. (How about the converse statement?)

Theorem 32. If a straight line XY parallel to BC meets the sides AB, AC of a triangle ABC at X and Y respectively, then $\Delta AXY \mathrel{|||} \Delta ABC$.

Proof. We have $\angle AXY = \angle ABC$ and $\angle AYX = \angle ACB$ being pairs of corresponding angles formed by the transversals AB, AC cutting the parallel lines XY and BC. Also

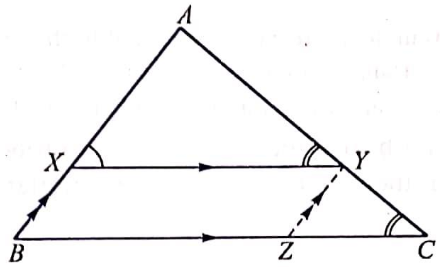


Fig. 3.62

$\angle XAY = \angle BAC$ (Fig. 3.62) Therefore the two triangles are *equiangular* in the sense that the angles of the one are respectively equal to the angles of the other.

Further $\dfrac{AX}{XB} = \dfrac{AY}{YC}$ since $XY \parallel BC$ (Theorem 29)

Therefore $\dfrac{AX}{AX + XB} = \dfrac{AY}{AY + YC}$ which gives $\dfrac{AX}{AB} = \dfrac{AY}{AC}$.

Through the point Y, draw $YZ \parallel AB$ meeting BC at Z. Then in ΔCAB, $YZ \parallel AB$ gives

$$\frac{AY}{AC} = \frac{BZ}{BC} = \frac{XY}{BC}.$$

since $BZ = XY$ in the parallelogram $BZYX$. Thus we have

$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC}.$$

Hence $\Delta AXY \mathrel{|||} \Delta ABC$. ❑

Theorem 33. Equiangular triangles are similar triangles.

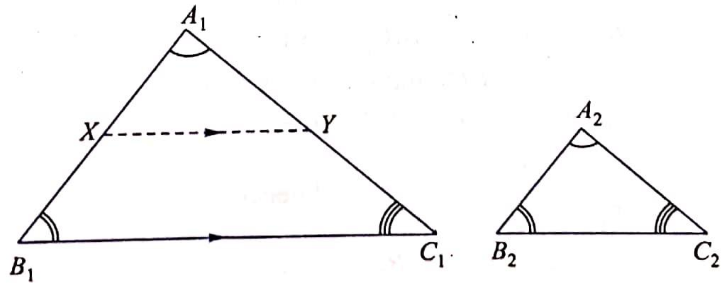


Fig. 3.63

Proof. Let the triangles $A_1B_1C_1$ and $A_2B_2C_2$ be equiangular. Then $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$ and $\angle C_1 = \angle C_2$. It is required to prove that the corresponding sides are proportional, *i.e.*,

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}$$

We may assume that $A_1B_1 \geq A_2B_2$. Let X be a point on A_1B_1 such that $A_1X = A_2B_2$ (Fig. 3.63). Draw $XY \parallel B_1C_1$ meeting A_1C_1 at Y. Now, $\angle A_1XY = \angle A_1B_1C_1$ (cor. angles). Therefore $\angle A_1XY = \angle A_2B_2C_2$. Also $\angle A_1YX = \angle A_1C_1B_1$ (cor. angles) implies that $\angle A_1YX = \angle A_2C_2B_2$. Further $A_1X = A_2B_2$ by our construction. Therefore the triangles A_1XY and $A_2B_2C_2$ are congruent and hence similar. By Theorem 32, $\Delta A_1XY ||| \Delta A_1B_1C_1$ and so we get $\Delta A_1B_1C_1 ||| \Delta A_2B_2C_2$ (See the observation (3) following the definition of similar triangles).

Note. If two angles of one triangle are respectively equal to the two corresponding angles of another triangle, then the two triangles are equiangular (*i.e.*, the third pair of angles also must be equal; this is true because the angles of a triangle add upto 180°) and hence similar.

Theorem 34. If two triangles have three sides of the one proportional to the three sides of the other, taken in order, then the two triangles are similar.

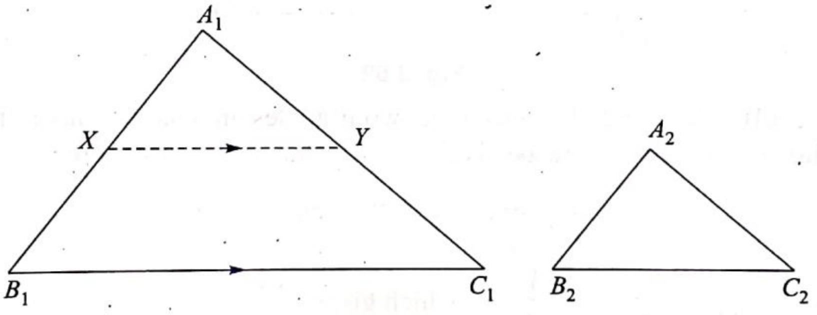


Fig. 3.64

Proof. In the triangles $A_1B_1C_1$ and $A_2B_2C_2$ we are given that

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}$$

It is required to prove that the two triangles are similar; in other words we have to prove that the corresponding angles of the two triangles are equal. We may assume without loss of generality that $A_1B_1 \geq A_2B_2$. Cut off A_1X on A_1B_1 such that $A_1X = A_2B_2$. Also cut off A_1Y on A_1C_1 such that $A_1Y = A_2C_2$.

We have $$\frac{A_1X}{A_1B_1} = \frac{A_2B_2}{A_1B_1} = \frac{A_2C_2}{A_1C_1} = \frac{A_1Y}{A_1C_1}.$$

Therefore by Theorem 29, $XY \parallel B_1C_1$ and so we see that

$$\Delta A_1XY ||| \Delta A_1B_1C_1 \text{ (Theorem 32)}$$

This gives $$\frac{A_1X}{A_1B_1} = \frac{A_1Y}{A_1C_1} = \frac{XY}{B_1C_1} \text{ and hence}$$

$$\frac{XY}{B_1C_1} = \frac{A_1X}{A_1B_1} = \frac{A_2B_2}{A_1B_1} = \frac{B_2C_2}{B_1C_1}$$

Therefore $XY = B_2C_2$.

Now by the *SSS* theorem, $\Delta A_1XY \equiv \Delta A_2B_2C_2$.

So, the two triangles A_1XY and $A_2B_2C_2$ are equiangular. Already, we have observed that $\Delta A_1XY \mathbin{|||} \Delta A_1B_1C_1$ and hence they are equiangular. Therefore $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ are equiangular and hence similar. ❑

Theorem 35. If two sides of one triangle are respectively proportional to two corresponding sides of the other and if the included angles are equal, then the two triangles are similar.

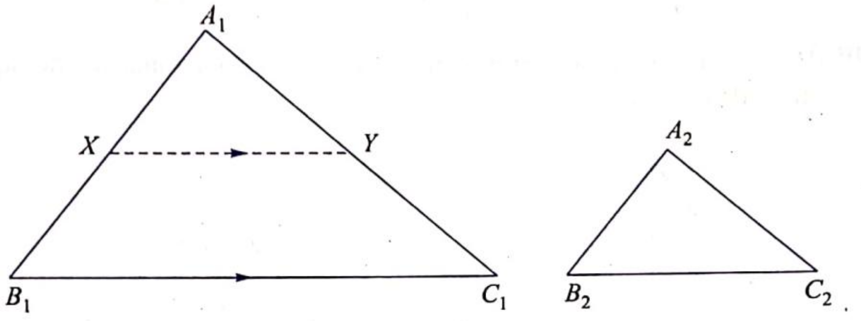


Fig. 3.65

Proof. We are given that $\dfrac{A_1B_1}{A_2B_2} = \dfrac{A_1C_1}{A_2C_2}$ and $\angle A_1 = \angle A_2$.

It is required to prove that the two triangles are similar. We may assume that $A_1B_1 \geq A_2B_2$. Let X be the point such that $A_1X = A_2B_2$. Draw $XY \parallel B_1C_1$ meeting A_1C_1 at Y. As before in Theorem 34, we have $\Delta A_1XY \mathbin{|||} \Delta A_1B_1C_1$ and hence

$$\frac{A_1X}{A_1B_1} = \frac{A_1Y}{A_1C_1} = \frac{XY}{B_1C_1}$$

But $A_1X = A_2B_2$ and therefore $\dfrac{A_2B_2}{A_1B_1} = \dfrac{A_1Y}{A_1C_1}$

Also, the hypothesis $\dfrac{A_1B_1}{A_2B_2} = \dfrac{A_1C_1}{A_2C_2}$ gives $A_1Y = A_2C_2$.

Therefore by the *SAS* test $\Delta A_1XY \equiv \Delta A_2B_2C_2$. Thus we have $\Delta A_1B_1C_1 \mathbin{|||} \Delta A_1XY$ and $\Delta A_1XY \equiv \Delta A_2B_2C_2$. This implies that $\Delta A_1B_1C_1 \mathbin{|||} \Delta A_2B_2C_2$. ❑

Theorem 36. In the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ we have $\angle A_1 = \angle A_2$. Then their areas are proportional to the rectangles contained by the sides containing $\angle A_1$ and $\angle A_2$.

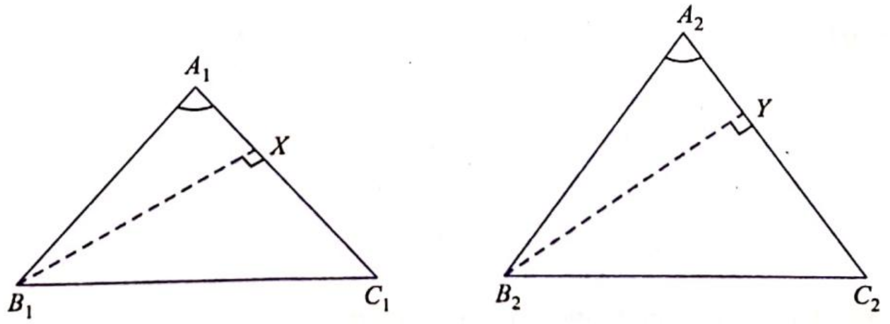


Fig. 3.66

Proof. Draw the altitudes B_1X and B_2Y (Fig. 3.66). Then the triangles A_1B_1X and A_2B_2Y are equiangular and hence similar. Therefore

$$\frac{A_1B_1}{A_2B_2} = \frac{A_1X}{A_2Y} = \frac{B_1X}{B_2Y} \qquad (*)$$

We have area of $\Delta A_1B_1C_1 = (1/2)\, A_1C_1 \,.\, B_1X$ and
area of $\Delta A_2B_2C_2 = (1/2)\, A_2C_2 \,.\, B_2Y$.

Therefore $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{A_1C_1 \,.\, B_1X}{A_2C_2 \,.\, B_2Y} = \frac{A_1C_1}{A_2C_2} \cdot \frac{A_1B_1}{A_2B_2} \quad \text{from}(*)$$ ❑

Theorem 37. The areas of two similar triangles are proportional to the squares on corresponding sides.

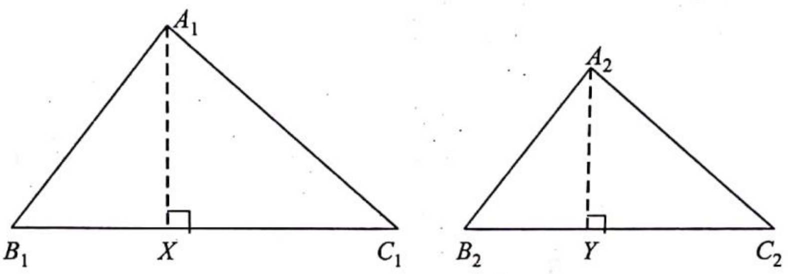


Fig. 3.67

Proof. Let $\Delta A_1B_1C_1$ be similar $\Delta A_2B_2C_2$. It is required to prove that

$$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1^2}{B_2C_2^2}$$

Draw the altitudes A_1X and A_2Y (Fig. 3.67).

Then $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1}{B_2C_2} \cdot \frac{A_1X}{A_2Y}$$

The triangles A_1B_1X and A_2B_2Y are equiangular and hence similar.

Therefore, $$\frac{A_1X}{A_2Y} = \frac{A_1B_1}{A_2B_2}; \quad \text{and} \quad \frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} \quad \text{(by hypothesis)}$$

This gives $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1}{B_2C_2} \cdot \frac{A_1X}{A_2Y} = \frac{B_1C_1^2}{B_2C_2^2}$$ ❑

Theorem 38. (Pythagoras's Theorem)

In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the sides containing the right angle.

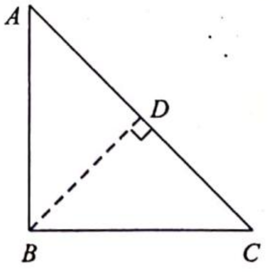


Fig. 3.68

Proof. The right triangles ABC, ADB and BDC are all similar to each other (Fig. 3.68). Therefore by Theorem 37

$$\frac{\Delta ABC}{AC^2} = \frac{\Delta ADB}{AB^2} = \frac{\Delta BDC}{BC^2}$$

This means that each of the ratios must be equal to

$$\frac{\Delta ADB + \Delta BDC}{AB^2 + BC^2} = \frac{\Delta ABC}{AB^2 + AC^2}$$

Hence $\quad AC^2 = AB^2 + BC^2.$

Note. The converse of Pythagoras's theorem is also true, namely "If in a ΔABC, $AC^2 = AB^2 + BC^2$ then ABC is a right triangle, right angled at B". The proof of this is left as an exercise.

Another proof of Pythagoras's theorem

Construct the squares $BCED$, $CAGF$ and $ABKH$ on the sides of the right triangle ABC, right angled at A. (Fig. 3.69). The triangles KBC and ABD are congruent (Why?).

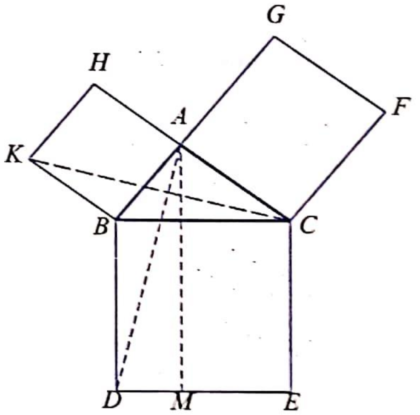


Fig. 3.69

$\therefore \quad \Delta KBC = \Delta ABD$ in area.

Now the square $KBAH$ and ΔKBC are on the same base KB and are between the same parallels.

$\therefore \quad$ Square $KBAH = 2\,\Delta KBC$ in area.

Similarly, $\quad$ Rectangle $BDMX = 2\Delta ABD$ and hence

$\quad$ Rectangle $BDMX = 2\Delta KBC =$ Sq. $KBAH$ (1)

Similarly, $\quad$ Rectangle $CEMX$ = Square ACFG (2)

Adding (1) and (2) we get

$$\text{Sq. } KBAH + \text{Sq. } ACFG = \text{Rect. } BDMX + \text{Rect. } CEMX = \text{Sq. } BCED.$$

Hence $\quad BC^2 = AB^2 + AC^2.$ ❑

EXERCISE 3.5

1. $ABCD$ is a parallelogram and E is the midpoint of CD. Prove that area of ΔADE = 1/4 the area of parallelogram $ABCD$.
2. Consider the family $\mathfrak{R}$ of parallelograms on equal bases whose areas are all equal. Prove that, in $\mathfrak{R}$ that which is a rectangle has the least perimeter.
3. On the line segment AB, parallelograms $ABCD$ and $ABXY$ are drawn on opposite sides and AB or AB produced bisects CX. Then prove that $ABCD$ and $ABXY$ have equal areas.
4. ABC is a fixed triangle. P is any point on the same side of BC as that of A such that ΔPBC and ΔABC have equal areas. Find the locus of P.
5. If the lengths of two sides of a triangle are given, show that its area is greatest when the angle between the sides is a right angle.

6. *ABCD* is a given parallelogram. What is the locus of the remaining vertices of a parallelogram equal in area to *ABCD*, having *AC* as a diagonal.
7. Let *ABCD* be a quadrilateral and *X* be the midpoint of *BD*. Prove that the area *AXCB* is one half the area of *ABCD*.
8. Let *ABCD* be quadrilateral. Suppose parallels are drawn through *A* and *C* to *BD* and through *B* and *D* to *AC*, prove that the resulting parallelogram has twice the area of *ABCD*.
9. If *D* is the midpoint of the side *BC* of ΔABC and *X* is any point on *AD*, prove that the triangles *AXB* and *AXC* are of equal area.
10. *ABC* is a triangle and *X* is any point such that area of ΔAXB= area of ΔXAC. Find the locus of *X*.
11. *ABCD* is a parallelogram and *X* is any point in the diagonal *AC*. Show that ΔABX and ΔADX are of equal area.
12. If two triangles have two sides of one equal to two sides of the other and the included angles are supplementary, prove that they are equal in area.
13. If in quadrilateral *ABCD*, *AC* bisects *BD*, show that *AC* also bisects the quadrilateral *ABCD*.
14. If *D* is a point on the side *AB* of ΔABC, find a point *X* on *BC* such that the triangles *XAD* and *CAX* are equal in area.
15. In quadrilateral *ABCD*, the diagonals *AC* and *BD* meet at *O*. Suppose the four triangles *AOB, BOC, COD* and *DO A* are equal in area, prove that *ABCD* is a parallelogram.
16. *D, E, F* are the midpoints of *BC, CA* and *AB* of ΔABC. Prove that ΔBCE and ΔBCF are of equal area; deduce that *FE* is parallel to *BC*.
17. In the family $\mathfrak{R}$ of all triangles on the same base whose areas are all equal, prove that the isosceles triangle in $\mathfrak{R}$ has the least perimeter.
18. If the diagonals of a quadrilateral *ABCD* meet at *O*, then prove that $\Delta ABC : \Delta ADC = BO : OD$.
19. In ΔABC, the straight lines *AD, BE, CF* are drawn through a point *P* to meet *BC, CA, AB* at *D, E, F* respectively. Prove that $PD/AD + PE/BE + PF/CF = 1$ and $AP/AD + BP/BE + CP/CF = 2$.
20. Let *ABCD* be a parallelogram and *P* be any point on *AC*. The line $XPY \parallel DA$ meets *DC* at *X* and *AB* at *Y*. Again, the line $QPR \parallel DC$ meets *AD* at *Q* and *BC* at *R*. Prove that $PX \cdot PQ = PY \cdot PR$.
21. In problem 20, take $AD = a$, $AB = b$, $XP = x$ and $QP = y$. Show that $x/a + y/b = 1$.
22. *ABCD* is a trapezium with $AB \parallel CD$. If the diagonals meet at *O*, prove that $AO : OC = BO : OD$.
23. *P* is a variable point on a given straight line and *A* is a fixed point. *Q* is a point on *AP* or *AP* produced such that $AQ : AP$ = constant. Find the locus of *Q*.
24. *ABC* is a triangle and *XY* is variable straight line parallel to *AC* meeting *BC* and *BA* in *X*, *Y* respectively. If *AX* and *CY* meet at *P*, find the locus of *P*.
25. In a quadrilateral *ABCD*, if the bisectors of $\angle A$ and $\angle C$ meet on *BD*, prove that the bisectors of $\angle B$ and $\angle D$ meet on *AC*.
26. In an isosceles ΔABC, the bisectors of the base angles *B* and *C* meet the opposite sides at *E* and *F* respectively. Prove that $FE \parallel BC$.
27. If A' is the midpoint of *BC* and if the internal bisectors of $\angle AA'B$ and $\angle AA'C$ meet *AB* and *AC* at *P* and *Q* respectively, prove that $PQ \parallel BC$.
28. The bisector of $\angle A$ in ΔABC meets *BC* at *U*. If *UX* is drawn parallel to *AC* meeting *AB* at *X*, and *UY* drawn parallel to *AB* meets *AC* at *Y*, prove that $BX/CY = AB^2/AC^2$.

29. ABC is a triangle right angled at A; AP and AQ meet BC or BC produced in P and Q and are equally inclined to AB. Show that $BP : BQ = PC : CQ$.

30. ABC is a triangle with $AB > AC$. The bisector of $\angle A$ meets BC at U and D is the midpoint of BC. Prove that $DU : DB = (AB - AC) : (AB + AC)$.

31. In ΔABC, BE and CF are the angular bisectors of $\angle B$ and $\angle C$ meeting at I. Prove that $AF/FI = AC/CI$.

32. If the bisector of $\angle A$ in ΔABC meets BC at D, prove that $BD = ac/(b + c)$ and $DC = ab/(b + c)$.

33. If the external bisector of $\angle A$ in ΔABC with $AB > AC$ meets BC produced at D' prove that $BD' = ac/(c - b)$ and $CD' = ab/(c - b)$.

34. With notations as in problems 32 and 33, prove that $DD' = 2abc/(c^2 - b^2)$.

35. In ΔABC, we have $AB > AC$. If A' is the midpoint of BC, AD is the altitude through A and if the internal and external bisectors of $\angle A$ meet BC at X and X' respectively, prove that

 (a) $A'X = a(c - b)/2(c + b)$

 (b) $A'X' = a(c + b)/2(c - b)$.

 (c) $A'D = (c^2 - b^2)/2a$.

36. (a) If the bisector of $\angle A$ in ΔABC meets BC at U, prove that $AU^2 = bc(1 - a^2/(b + c)^2)$.

 (b) If the external bisector meets BC at U' then prove that $AU'^2 = bc(a^2/(c - b)^2 - 1)$.

37. $ABCD$ is a parallelogram. The side CD is bisected at P and BP meets AC at X. Prove that $3AX = 2AC$.

38. $ABCD$ is a parallelogram. X divides AB in the ratio 3 : 2 and Y divides CD in the ratio 4 : 1. If XY cuts AC at Z, prove that $7AZ = 3AC$.

39. $ABCD$ is a trapezium with $AB \parallel CD$ and $AB = 2CD$. If the diagonals meet at O, then prove that $3AO = 2AC$. If AD and BC meet at F, then prove that $AD = DF$.

40. OA, OB, OC are three given line segments and P is any point on OC. If PM and PN are the perpendiculars from P on OA and OB respectively, prove that $PM : PN$ is a constant.

41. OA_1, OA_2, OA_3 are three given straight lines. Two parallel straight lines AB and CD cut OA_1, OA_2, OA_3 at P, Q, R and P', Q', R' respectively. Prove that $PQ : QR = P'Q' : Q'R'$.

42. $ABCD$ is a trapezium with $AB \parallel CD$ and the diagonals meet at O. If $XOY \parallel AB$ meets AD and BC at X and Y then prove that $XO = OY$.

43. If ABC and DEF are similar triangles and if AX and DY are the altitudes of the triangles, through A and D, prove that $AX : DY = BC : EF$.

44. $ABCD$ is a parallelogram, and AXY is a straight line through A meeting BC at X and DC at Y. Prove that $BX.\ DY$ is a constant.

45. $ABCD$ is a parallelogram. A straight line through A meets BD at X, BC at T and DC at Z. Prove that $AX : XZ = AY : AZ$.

46. ABC is a triangle and DAE is a straight line parallel to BC such that $DA = AE$. If CD meets AB at X and BE meets AC at Y, prove that $XY \parallel BC$.

47. Given four points A, B, C, D in a straight line, find a point O in the same straight line such that $OA : OB = OC : OD$.

48. $ABCD$ and $AECF$ are two parallelograms and side EF is parallel to AD. Suppose AF and DE meet at X and BF, CE meet at Y, then prove that $XY \parallel AB$.

49. If in triangles ABC and DEF, we have $\angle A = \angle D$ and $AB : DE = BC : EF$. Then prove that $\angle C = \angle F$ or $\angle C + \angle F = 180°$.

50. In a right angled triangle ABC with $\angle A = 90°$, if AD is the altitude, from A on BC, then triangles DBA and DAC are each similar to ΔABC. A also prove DA is a mean proportional to DB and DC.

51. If ΔABC is similar to ΔDEF with X and Y as midpoints of corresponding sides BC and EF, then prove that $AX : DY = BC : EF$.

52. We are given that ΔABC is similar to ΔDEF with X and Y dividing BC and EF in the same ratio. Prove that ΔABX is similar ΔDEY.

53. P is any point within ΔABC. Q is a point outside ΔABC such that $\angle CBQ = \angle ABP$ and $\angle BCQ = \angle BAP$. Show that the triangles PBQ and ABC are similar.

54. In two obtuse angled triangles, an acute angle of the one is equal to an angle of the other; and also the sides about the other acute angles are proportional.

Prove that the triangles are similar.

55. PM and PN are the perpendiculars from a point to two given straight lines OA and OB. If PM/PN is a constant, prove that the locus of P is a straight line through O.

56. From A, perpendiculars AX, AY are drawn to the bisectors of the exterior angles of B and C of ΔABC. Prove that $XY \parallel BC$.

57. If ABC and XYZ are two triangles such that $AB : BC = XY : YZ$ and the angles B and Y are supplementary prove that $\{\Delta ABC/\Delta XYZ\} = AB^2/XY^2$.

58. A straight line is drawn through the incentre I of ΔABC, perpendicular to AI meeting AB, AC in D and E respectively. Prove that $BD.\ CE = ID^2$.

59. AD bisects $\angle A$ of ΔABC and meets BC at D. If S and S' are the circumcentres of ΔABC, show that $SD/S'D = BD/DC$.

60. If N is a point on the straight line AB and $PN \perp AB$ then prove that $AP^2 - BP^2 = AN^2 - BN^2$.

61. If in quadrilateral $ABCD$, $AC \perp BD$, show that $AB^2 + CD^2 = BC^2 + DA^2$.

62. Let ABC be an equilateral triangle and AD be the altitude through A. Show that $AD^2 = 3BD^2$.

63. In a given straight line AB, find a point P such that the difference in the squares on AP and PB is equal to the difference between two given squares.

64. ABC is a right angled triangle right angled at A. AD is the altitude through A; E is a point on AC such that $AE = CD$ and F is a point on AB such that $AF = BD$. Prove that $BE = CF$.

3.6 CONCURRENCE AND COLLINEARITY

Three points A, B, C are *collinear* if they all lie on a straight line. If A, B, C are as shown in Figure 70, then $AB + BC = AC$. If we use directed segments, we always have

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0.$

Fig. 3.70

Theorem 39. (Stewart's theorem)

If A, B, C are three collinear points and P any other point, then

$$PA^2 \,.\, BC + PB^2 \,.\, CA + PC^2 \,.\, AB + BC \,.\, CA \,.\, AB = 0.$$

(using directed segments).

Case. (*i*) P does not lie on the straight line ABC.

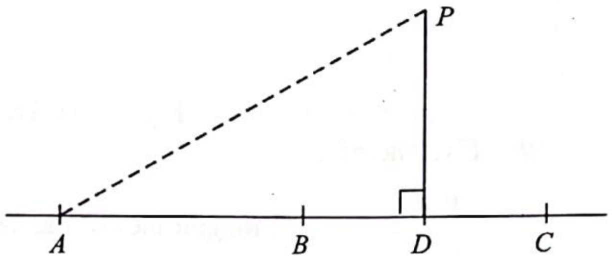


Fig. 3.71

Proof. Let D be the foot of the perpendicular from P on the line ABC. Then we have

$$PA^2 = PD^2 + DA^2 = PD^2 + (DC + CA)^2$$
$$= PD^2 + DC^2 + CA^2 + 2DC \,.\, CA.$$
$$PB^2 = PD^2 + DB^2 = PD^2 + (DC + CB)^2$$
$$= PD^2 + DC^2 + CB^2 + 2DC \,.\, CB.$$

Now $PC^2 = PD^2 + DC^2$ and hence we get

$$PA^2 = PC^2 + CA^2 + 2DC \,.\, CA.$$
$$PB^2 = PC^2 + CB^2 - 2DC \,.\, BC \text{ (since } CB = -BC\text{)}.$$

This gives

$$PA^2 \,.\, BC + PB^2 \,.\, CA = PC^2 \,.\, BC + CA^2 \,.\, BC + 2DC \,.\, CA \,.\, BC + PC^2 \,.\, CA + CB^2 \,.\, CA - 2DC \,.\, BC \,.\, CA$$
$$= PC^2(BC + CA) + (BC \,.\, CA)(BC + CA).$$
$$= (PC^2 + BC \,.\, CA)\,(BA).$$
$$= (PC^2 + BC \,.\, CA)\,(-AB).$$

Hence $PA^2 \,.\, BC + PB^2\, CA + PC^2 \,.\, AB + BC.\, CA \,.\, AB = 0.$

Case. (*ii*) P lies on the straight line ABC.

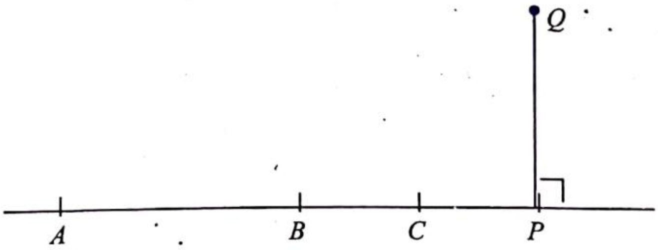


Fig. 3.72

Let Q be any point on the perpendicular through P to the line ABC. Then by case (*i*) we have

$$QA^2 \cdot BC + QB^2 \,.\, CA + QC^2 \,.\, AB + BC \,.\, CA \,.\, AB = 0.$$

We note that $QA^2 = QP^2 + PA^2$, $QB^2 = QP^2 + PB^2$ and $QC^2 = QP^2 + PC^2$.

Substituting in the above equation we get

$$QP^2(BC + CA + AB) + PA^2 \,.\, BC + PB^2 \,.\, CA + PC^2 \,.\, AB + BC \,.\, CA \,.\, AB = 0.$$

Now $BC + CA + AB = 0$ and hence we get our required result. ❑

Theorem 40. (Menelaus's Theorem) If a transversal cuts the sides BC, CA, AB of a triangle ABC at D, E, F respectively then

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$$

Proof. Draw $CX \parallel BA$ meeting the transversal at X (Fig. 3.73). The triangles FBD and XCD are similar since $BF \parallel CX$. Therefore,

$$\frac{BD}{DC} = \frac{FB}{CX} \text{ (in magnitudes of the segments).}$$

Again the triangles EAF and ECX are similar and so

$$\frac{CE}{EA} = \frac{CX}{AF} \text{ (in magnitudes of the segments)}$$

Hence we get $$\frac{BD}{DC}\cdot\frac{CE}{EA} = \frac{FB}{CX}\cdot\frac{CX}{AF}$$

Therefore $$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \text{ in magnitude.}$$

Now, we shall examine the sign of the product. A transversal cuts two sides internally and the other side externally as in Fig. 3.73 or cuts all the three sides externally as in Fig. 3.74.

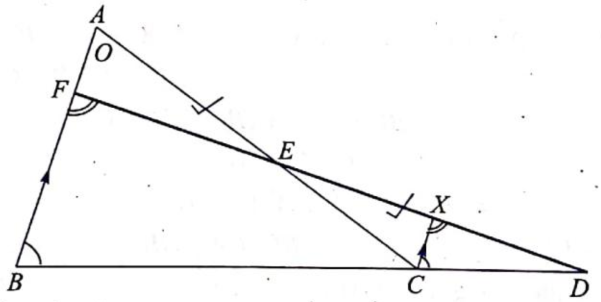


Fig. 3.73

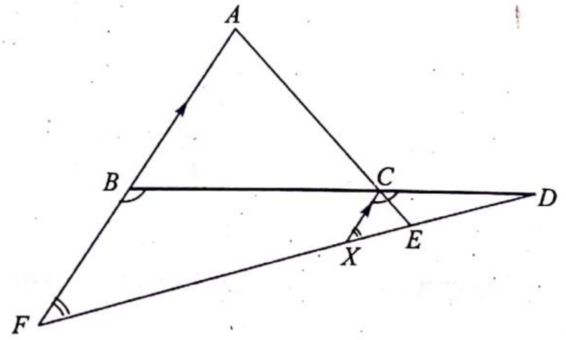


Fig. 3.74

In the former case (as in Fig. 3.73) we have one ratio negative and the other two positive. In our Fig. 3.73,

$$\frac{BD}{DC} < 0; \quad \frac{CE}{EA} \text{ and } \frac{AF}{FB} \text{ are positive.}$$

In the latter case (as in Fig. 3.74), all the ratios

$\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}$ are negative. Thus the product

$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}$ is always negative and

we have $$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$$ ❑

Theorem 41. If D, E, F are points on the sides BC, CA, AB of a ΔABC such that

$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$ then D, E, F are collinear.

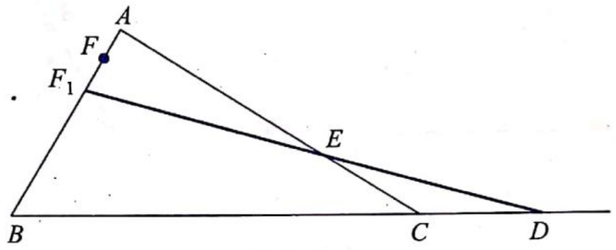


Fig. 3.75

Proof. Suppose DE meets AB at F_1. Then by Menelaus's theorem

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF_1}{F_1B} = -1.$$

This along with our hypothesis

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$$

implies that $$\frac{AF}{FB} = \frac{AF_1}{F_1B}.$$

This means that F and F_1 must coincide with each other (Theorem 30). Therefore D, E and F are collinear. ❑

Theorem 42. (Ceva's Theorem)

If the lines joining the vertices A, B, C of a triangle ABC to any point S in their plane meet the opposite sides in D, E, F then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1.$$

Proof. If S lies inside ΔABC, then all the three points D, E, F divide the corresponding sides internally. If S is outside ΔABC, then two of the three points D, E, F divide the corresponding sides externally while the remaining point divides the corresponding side internally. (Fig. 3.76(a) and Fig. 3.76(b)).

Draw BX and CY parallel to ASD meeting BS and CS at Y and X respectively. The triangles AFS and BFX are similar. Therefore

$$\frac{AF}{FB} = \frac{AS}{BX} \quad (1)$$

$$\Delta XBC \,|||\, \Delta SDC \quad \text{gives} \quad \frac{BD}{DC} = \frac{XB}{SD} \tag{2}$$

$$\Delta BDS \,|||\, \Delta BCY \quad \text{gives} \quad \frac{BD}{BC} = \frac{SD}{YC} \tag{3}$$

$$\Delta CEY \,|||\, \Delta AES \quad \text{gives} \quad \frac{CE}{EA} = \frac{CY}{AS} \tag{4}$$

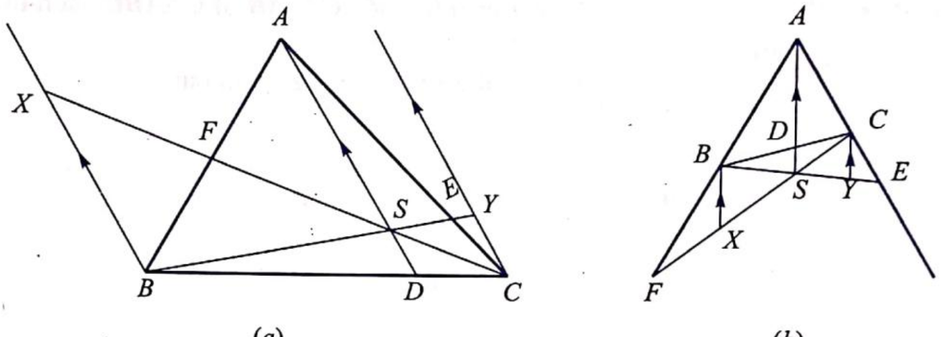


Fig. 3.76

In all these ratios only the lengths of the segments are taken into account. We will examine the signs later. Multiplying the above four equations we get

$$\frac{AF}{FB} \cdot \frac{BC}{DC} \cdot \frac{BD}{BC}\frac{CE}{EA} = \frac{AS}{BX} \cdot \frac{XB}{SD} \cdot \frac{SD}{YC} \cdot \frac{CY}{AS} = 1$$

This gives $\quad \dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = 1$

When S lies inside, all these ratios $\dfrac{BD}{DC}, \dfrac{CE}{EA}, \dfrac{AF}{FB}$ are positive; and when S lies outside ΔABC one ratio is positive and the other two are negative. In Fig. 3.76(b), $BD/DC > 0$, $CE/EA < 0$, $AF/FB < 0$.

Thus, $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = +1$ (both in magnitude and sign).

Second Proof. Consider ΔADC and the transversal BSE. By Menelaus' theorem

$$\frac{DB}{BC} \cdot \frac{CE}{EA} \cdot \frac{AS}{SD} = -1 \tag{1}$$

Again from ΔADB and the transversal CSX, we get

$$\frac{BC}{CD} \cdot \frac{DS}{SA} \cdot \frac{AF}{FB} = -1 \tag{2}$$

Multiplying (1) and (2) we get

$$\frac{DB}{CD} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1$$

Equivalently $\quad \dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = +1$

Third Proof :

We have $\dfrac{BD}{DC} = \dfrac{\Delta ABD}{\Delta ADC} = \dfrac{\Delta SBD}{\Delta SDC} = \dfrac{\Delta ABD - \Delta SBD}{\Delta ADC - \Delta SDC} = \dfrac{\Delta ABS}{\Delta CAS}$

Similarly, $\dfrac{CE}{EA} = \dfrac{\Delta BCS}{\Delta ABS}$ and $\dfrac{AF}{FB} = \dfrac{\Delta CAS}{\Delta BCS}$.

Multiplying we get $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = +1.$ ❑

Theorem 43. (Converse of Ceva's Theorem)

If three Cevians *AD*, *BE*, *CF* satisfy

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

then they are concurrent.

[By a *Cevian* we mean a line segment joining a vertex of a triangle to any given point on the opposite side].

Proof. Suppose the two Cevians *AD* and *BE* meet at *H*. Let the Cevian through C and H be CF_1. Then by Ceva's theorem

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF_1}{F_1B} = 1$$

By assumption $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = 1$

Therefore $\dfrac{AF}{FB} = \dfrac{AF_1}{F_1B}$ and hence $F = F_1$.

This proves that *AD*, *BE CF are* concurrent. ❑

Corollary. (1) If *D*, *E*, *F* are the midpoints of *BC*, *CA*, *AB* then the medians *AD*, *BE*, *CF* are concurrent.

(2) If the Cevians *AD*, *BE*, *CF* are the internal bisectors of the angles of the *ABC*, then *AD*, *BE*, *CF* are concurrent.

(3) If the Cevians *AD*, *BE*, *CF* are the altitudes of the *ABC*, then they are concurrent

Proof. (1) If *D, E, F* are the midpoints of *BC*, *CA*, *AB* then we have

$$\frac{BD}{DC} = 1 = \frac{CE}{EA} = \frac{AF}{FB} \text{ and hence } \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Therefore, *AD*, *BE* and *CF* are concurrent.

(2) If *AD*, *BE*, *CF* are the angular bisectors of the angles of $\Delta\, ABC$,

then $\dfrac{BD}{DC} = \dfrac{AB}{AC}, \dfrac{CE}{EA} = \dfrac{BC}{BA}$ and $\dfrac{AF}{FB} = \dfrac{CA}{CB}$

Therefore $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = 1.$

Hence the internal bisectors of the angles of a triangle are concurrent. (Note that, here all the ratios are positive).

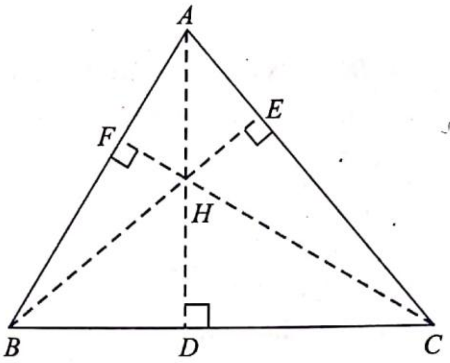


Fig. 3.77

(3) $\Delta ABD \,|||\, \Delta CBF$ gives $\dfrac{BD}{FB} = \dfrac{AD}{CF}$ (1)

$\Delta ADC \,|||\, \Delta BEC$ gives $\dfrac{AD}{DC} = \dfrac{BE}{EC}$ (2)

$\Delta AFC \,|||\, \Delta AEB$ gives $\dfrac{AF}{EC} = \dfrac{AE}{EB}$ (3)

Multiplying the three equations we get $\dfrac{BD}{FB} \cdot \dfrac{AD}{DC} \cdot \dfrac{AF}{FC} = \dfrac{AD}{CF} \cdot \dfrac{BE}{EC} \cdot \dfrac{AE}{EB}$

Therefore $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = 1$ (in magnitude)

Again, all the ratios $\dfrac{BD}{DC}, \dfrac{CE}{EA}, \dfrac{AF}{FB}$ positive and hence

AD, BE and CF are concurrent. ❑

EXERCISE 3.6

1. Let ABC be any triangle right angled at A. If D, E trisect the hypotenuse BC then prove that $AD^2 + AE^2 = 5/9BC^2$.
2. If from a point O, OD, OE, OF are drawn perpendicular to the sides BC, CA, AB respectively of ΔABC then prove that
$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0.$$
3. The median AA' of the ΔABC meets the side $B'C'$ of the medial triangle $A'B'C'$ in P and CP meets AB in Q. Show that $AB = 3\,AQ$.
4. If a line through the centroid G of ΔABC meets AB in M and AC in N then prove that $AN.\ MB + AM.\ NC = AM.\ AN$ both in magnitude and sign.
5. Prove that the triangle formed by the points of contact of the sides of a given triangle with the excircles corresponding to these sides is equivalent to the triangle formed by the points of contact of the sides of the triangle with the inscribed circle.
6. (*i*) Prove that the external bisectors of the angles of a triangle meet the opposite sides in three collinear points.

   (*ii*) Prove that the internal bisectors and the external bisector of the third angle meet the opposite sides in three collinear points.
7. Prove that the sides of the orthic triangle meet the sides of the given triangle in three collinear points.

8. B' and C' are the midpoints of AC, AB of ΔABC; Q is the midpoint of $B'C'$ and BQ meets AC at R. Prove that $AR/RC = 1/2$.
9. If H is any point within ΔABC, prove that the external bisectors of the angles AHB, BHC, CHA meet AB, BC, CA respectively at three collinear points.
10. Points X and Y are taken on the sides CA, AB of ΔABC such that $CX/XA = AY/YB = \lambda$. If XY and CB produced meet at D prove that $CD = \lambda^2 BD$.
11. G is the centroid of ΔABC; AG is produced to X such that $GX = AG$. If we draw parallels through X to CA, AB, BC meeting BC, CA, AB at L, M, N respectively, prove that L, M, N are collinear.
12. In ΔABC, XY is drawn parallel to BC cutting AB, AC in X and Y. Prove that BY and CX intersect on the median through A.
13. AD, BE, CF are concurrent lines in a ΔABC. Show that the lines through the midpoints of BC, CA, AB respectively parallel to AD, BE, CF are concurrent.
14. Equilateral triangles DBC, ECA and FAB are constructed externally on the sides BC, CA, AB of ΔABC. Prove that AD, BE and CF are also concurrent.
15. In ΔABC, AD, BE and CF are concurrent lines. P, Q, R are points on EF, FD, DE such that DP, EQ and FR are concurrent. Prove that AP, BQ and CR are also concurrent.
16. AD, BE and CF are three concurrent lines in ΔABC, meeting the opposite sides in D, E, F respectively. Show that the joins of the midpoints of BC, CA, AB to the midpoints of AD, BE and CF are concurrent.
17. AD, BE and CF are three concurrent lines meeting the sides BC, CA, AB of ΔABC in D, E, F respectively. Suppose EF, FD and DE meet BC, CA, AB at X, Y and Z respectively. Prove that B, C divide DX harmonically and that X, Y, Z are collinear.
18. If X and Y are variable points on the sides CA, AB of ΔABC such that $CX/XA + AB/AY = 1$, prove that XY passes through a fixed point.
19. A straight line cuts the sides AB, BC, CD, DA of a quardrilateral in P, Q, R, S. Prove that $AP/PB \,.\, BQ/QC \,.\, CR/RD \,.\, DS/SA = 1$.
20. If two triangles ABC and DEF are such that the perpendiculars from A, B, C to EF, FD, DE are concurrent, prove that the perpendiculars from D, E, F to BC, CA, AB are concurrent.
21. In quadrilateral $ABCD$, let AB and CD meet at E and AD and BC meet at F. Then prove that the midpoints of AC, BD and EF are collinear.
22. Four points P, Q, R, S are taken on the sides AB, BC, CD, DA of a quadrilateral such that $AP/PB \,.\, BQ/QC \,.\, CR/RD \,.\, DS/SA = 1$. Prove that PQ and RS intersect on AC.
23. ABC and $A'B'C'$ are two triangles such that AA', BB', CC' meet at O. Prove that if BC, $B'C'$ meet at L; CA, $C'A'$ meet at M; AB, $A'B'$ meet at N then L, M, N are collinear.

PROBLEMS

1. $ABCD$ is a quadrilateral such that the sum of the angles at A and B is equal to the sum of the angles at C and D. Prove that two sides of the quadrilateral are parallel to one another.
2. Draw four straight lines at random.
   (a) In how many points do the lines intersect?
   (b) How many triangles are formed?
   (c) Into how many regions do the lines divide the plane?
3. A triangle is rotated in its own plane about the point A into a position $A'B'C'$. If AC bisects BB', prove that AB' bisects CC'.

4. If $ABCD$ is a quadrilateral in which $AB + CD = BC + AD$, prove that the bisectors of the angles of the quadrilateral meet in a point which is equidistant from the sides of the quadrilateral.
5. $ABCDEFGH$ is a regular octagon and AF, BE, CH, DG are drawn. Prove that their intersections are the angular points of a square.
6. A figure consists of five equal squares in the form of a cross. Show how to divide it by two straight cuts into four equal figures which will fit together to form a square.
7. $ABCD$ is a quadrilateral and X is a given point in AD. Find a point Y in AB such that the area of the ΔAXY is equal to that of $ABCD$. Hence show how to divide the quadrilateral $ABCD$ into three equal parts by straight lines drawn through X.
8. A square of perimeter 52 is inscribed in a square of perimeter 68. What are the possible distances from a fixed vertex of the inner square to the four vertices of the outer square.
9. Given a hexagon of side $2a$ and 25 points inside it, show that there are at least two points among them whose distance apart is at most a units.
10. Let X be a point inside a rectangle $ABCD$. If $XA = a$, $XB = b$, $XC = c$. Find XD.
11. In ΔABC, find points X, Y, Z on AB, BC, CA such that $AXYZ$ is a rhombus. Show that the area of the rhombus $AXYZ \le (1/2)\ \Delta ABC$.
12. Equilateral triangles BCX, CAY and ABZ are constructed externally on the sides of ΔABC. I, P, Q, R are the midpoints of BX, BZ and AC prove that ΔPQR is equilateral.
13. Given a parallelogram $ABCD$, a straight line cuts off 1/3 AB, 1/4 and λAC from the segments AB, AD and AC respectively. Find λ.
14. The bisector of each angle of a triangle intersects the opposite side at a point equidistant from the midpoints of the other two sides of the triangle. Find all such triangles.
15. $ABCD$ is a parallelogram in which $AB/BC = \lambda$. If P, Q are points on the line CD with P on CD and M is a point on BC such that AD bisects $\angle PAQ$ and AM bisects $\angle PAB$, $BM = a$, $DQ = b$ prove that $AQ = a/\lambda + b$.
16. In ΔABC a point X is taken on AC and a point Y is taken on BC. If AY and BX meet at O, find the area of ΔCXY if the areas of triangles OXA, OAB and OBX are x, y, z respectively.
17. $ABCD$ is a trapezium with AD || BC; $AD = 3BC$ and a transversal XY cuts BC at X and AD at Y. If EF is a line segment contained in XY such that $AE \parallel DF$. $BE \parallel CF$ and $AE/DF = CF/BE = 2$, show that the area of ΔEFD = 1/4 area $ABCD$.
18. In ΔABC, X and Y are points on the sides AC and BC respectively. If Z is on the segment XY such that $\dfrac{AX}{XC} = \dfrac{CY}{YB} = \dfrac{XZ}{ZY}$ prove that the area of ΔABC is given by $\Delta ABC = \{(\Delta AXZ)^{1/3} + (\Delta BYZ)^{1/3}\}^3$.
19. $ABCD$ is a trapezium with $AD \parallel BC$; X line on AD such that $AX/XD = \lambda$. The straight lines AB and CD meet at E, and the lines BX and AC meet at F. If EF meets AD at Y prove that $AY/YD = \lambda/(\lambda + 1)$.
20. ABC is a triangle and A_1, B_1, C_1 are points on BC, CA, AB such that

$$BA_1/A_1C = CB_1/B_1A = AC_1/C_1B = \lambda.$$

If A_2, B_2, C_2 are points on B_1C_1, C_1A_1, and A_1B_1 such that

$$B_1A_2/A_2C_1 = C_1B_2/B_2A_1 = A_1C_2/C_2B_1 = 1/\lambda,$$

prove that $\Delta\, ABC$ is similar to $\Delta A_2B_2C_2$ and find the ratio of similitude.
21. In ΔABC, D, E, F are points on the sides BC, CA, AB. Also, A, B, C are points on YZ, ZX, XY of ΔXYZ for which $EF \parallel YZ$, $FD \parallel ZX$, $DE \parallel XY$. Prove that area of $\Delta\, ABC = \{\text{area } \Delta\, DEF.\ \text{area } \Delta\, XYZ\}^{1/2}$.

22. l and m are two straight lines intersecting at O. If the perpendiculars from A to l, m meet the straight lines l, m at X, Y respectively; and the perpendiculars from B to l, m meet them at P, Q show that the angle between XY and PQ is $\angle AOB$ (assume that $\angle AOB$ is acute).

23. Prove that the locus of the point P which moves such that $AM^2 - MB^2 = \lambda$ = constant is a straight line AB, where A and B are two fixed points.

24. O is a point in the plane of ΔABC with $OA = x$, $OB = y$ and $OC = z$. Prove that there is no $d > o$ and no point P such that $PA = \sqrt{(x^2 + d)}$, $PB = \sqrt{(y^2 + d)}$ and $PC = \sqrt{(z^2 + d)}$.

25. ABC is a triangle P is a point inside ΔABC such that its distances from the sides of ΔABC are x, y, z. If a, b, c, k are given constants, prove that the locus of P such that $ax + by + cz = k$ is either an empty set or a line segment or coincides with the set of all points in ΔABC.

26. Find the locus of points P within a given ΔABC and such that the distances from P to the sides of the given triangle can themselves be the sides of a certain triangle.

27. Let/be a fixed straight line and l_1, l_2, l_3 be three lines perpendicular to l. A, B, C are three fixed points on l; A_1, B_1, C_1 are three points on l_1, l_2, l_3 respectively such that the perpendiculars from A, B, C to B_1, C_1, C_1A_1 and A_1B_1 meet at a point. Then prove that for any three points X, Y, Z on l_1, l_2, l_3 respectively, the perpendiculars from A, B, C to YZ, ZX, XY meet at a point.

28. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the other two angles are collinear.

29. The vertex A of a variable ΔABC is fixed; the side BC of variable length lies on a fixed straight line. Show that the locus of the projection of A upon the bisectors of $\angle B$ and $\angle C$ is a straight line.

30. Show that the sum of the reciprocals of the internal bisectors of a triangle is greater than the sum of the recipocals of the sides of the triangle.

31. The internal bisector of the $\angle B$ of ΔABC meets the sides $B'C'$ and $B'A'$ of the medial triangle in the points A'', C'' respectively. Prove that AA'', CC' are perpendicular to the bisector of $\angle B$ and that $B'A'' = B'C''$. (Similar result holds for external bisector).

32. In any ΔABC if L, M, N are three collinear points on the sides BC, CA, AB and L', M', N' are points on BC, CA, AB respectively such that $A'L = A'L'$, $B'M$, $C'N = C'N'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that L', M', N' are also collinear. (Two such transversals LMN, $L'M'N'$ are called *reciprocal* or *isotomic* transversals).

33. Draw two straight lines of given directions so that they are reciprocal transversals for a given triangle.

34. If D, D′; E, E'; F, F' are isotomic pairs of points on the sides BC, CA, AB of ΔABC then prove that area ΔDEF = area $\Delta D'E'F'$.

35. Two equal segments AE, AF are taken on the sides AB, AC, of a ΔABC. Show that if the median through A of ΔABC meets EF at X then $EX/XF = AC/CB$.

36. D, E, F are any three points on BC, CA, AB of ΔABC such that AD, BE, CF are concurrent at P. If D', E', F' are the isotomics of D, E, F then prove that AD', BE', CF' are also concurrent. (If we call this point as Q then P and Q are called *isotomic conjugates*; see problem 33).

37. If AD, BE, CF are Cevians of ΔABC concurring at a point O, then prove that

$$OD/AD + OE/BE + OF/CF = 1.$$

38. With notations as in 38, prove that

$$AO/OD = AE/EC + AF/FB.$$

39. If two triangles are symmetrical with respect to a poin, show that the reciprocal transversals of the sides of one triangle with respect to the other are concurrent.

40. Show that the lines joining the vertices of a given equilateral triangle to the images of a given point in the respectively opposite sides are concurrent.

41. Prove that the lines joining the midpoints of three concurrent Cevians to the midpoints of ths corresponding sides of the given triangle are concurrent.

42. If O, A, C, D are collinear and $OA^2 = OC \cdot OD$ both in magnitude and sign, then prove that the symmetric of A with respect to O is the harmonic conjugate of A with respect to C, D.

43. If CD divide AB harmonically and O is the midpoint of AB prove that

$$OC^2 + OD^2 = CD^2 + 2OA^2.$$

44. If C, D divide AB harmonically and A', B' are the harmonic conjugates of D with respect to the pairs A, C and B, C prove that C, D also divide $A' B'$ harmonically.

45. If C, D divide AB harmonically and O is he midpoint of AB prove that

(*i*) $1/(CA \cdot CB) + 1/(DA\ DB) = 1/(OA \cdot OB)$.

(*ii*) $1/BC = 1/AB + 1/AD + 1/CD$.

(*iii*) $DA \cdot DB = DC\ .\ DO$.

46. Let AB be divided harmonically at C and D . O is a point not on AB and the straight line through B parallel to OA meets OC, OD at P, Q respectively. Prove that $PB = BQ$.

47. If A, B, C, D are collinear points and O is a point not on AB such that the parallel through B to OA meets OC, OD at P, Q with $PB = BQ$, then prove that C, D divide AB harmonically.

48. If AB is divided harmonically by C, D and O is a point not on AB, then prove that any transversal cuts OA, OB, OC, OD in four harmonic points.

49. If C, D divide AB harmonically and C', D' divide $A' B'$ harmonically; and the lines AA', BB', CC', meet at a point O, then prove that DD' also passes through O.

50. A straight line through A of the parallelogram $ABCD$ meets BD, BC, CD in E, F, G respectively. Show that $1/AE = 1/AF + 1/AG$.

51. A straight line $PGQR$ through the centroid G of ΔABC meets AB, AC and BC produced at P, Q, R respectively. Prove that $1/GP = 1/GQ + 1/GR$.

52. Consider ΔABC and a point within the triangle. If AP, BP, CP meet the opposite sides in D, E, F respectively, prove that of the numbers $AP/PD, BP/PE, CP/PF$ at least one is less than or equal to 2 and at least one is greater than or equal to 2.

53. AOB is a triangle with $\angle AOB < 90°$. Through a point $M \neq O$, perpendiculars are drawn to OA and OB meeting OA, OB at P, Q respectively. H is the orthocentre of ΔOPQ. What is the locus of H if M is permitted to range over (*i*) the side AB (*ii*) the interior of ΔOAB?

54. P, Q, R are points on the sides BC, CA, AB of ΔABC. Prove that the area of at least one of the triangles AQR, BRP, CPQ is less than or equal to one quarter of the area of ΔABC.

55. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the mid points of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are twelve vertices of a regular dodecagon.

56. Let A_1A_2, B_1B_2, C_1C_2 be three equal segments on the sides of a equilateral triangle ABC. Prove that in the triangle formed by the lines B_2C_1, C_2A_1, A_2B_1 the segments B_2C_1, C_2B_1 are proportional to the sides in which they are contained.

57. ABC is a triangle and the line YCX is parallel to AB such that AX and BY are the angular bisectors of $\angle A$ and $\angle B$ respectively. If AX meets BC at D and BY meets AC at E and if $YE = XD$ prove that $AC = BC$.

58. Let ABC be an equilateral triangle with side a. M is a point such that $MS = d$ where S is the centre of ΔABC. Prove that the area of the triangle whose sides are of length MA, MB, MC is $[\sqrt{3}/12]\ |a^2 - 3d^2|$.

59. Give two equilateral triangles $A_1B_1C_1$ and $A_2B_2C_2$, find the locus of points M such that the two triangles formed by the line segments MA_1, MB_1, MC_1 and the segments MA_2, MB_2, MC_2 are of the same area.

Chapter 4 Geometry Circles Page 86

CHAPTER

4

GEOMETRY–CIRCLES

4.1 CIRCLES-PRELIMINARIES

A circle is a geometric figure in a plane such that all its points are equidistant from a fixed point in the plane. The fixed point is the centre of the circle and the constant distance from the centre is the radius of the circle.

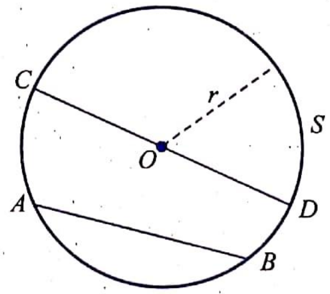


Fig. 4.1

The circle S in Fig. 4.1 has centre O and radius r.

A chord of a circle is a straight line segment joining any two points on the circle. A chord passing through the centre is called a *diameter* of the circle. In Fig. 4.1, AB is a chord of the circle S and CD is a diameter. Circles which have the same centre are called *concentric circles*.

We observe that a point is within, upon or outside a circle according as its distance from the centre is less than, equal to, or greater than the radius. Concentric circles whose radii are unequal, do not intersect with each other. A circle is symmetric about any of its diameters (Fig. 4.2). Also, a circle is symmetric about its centre (Fig. 4.3).

Theorem 1. The perpendicular bisector of any chord of a circle passes through the centre of the circle.

Proof. Let O be the centre of a circle S and AB be any chord; let C be the mid point of AB (See Fig. 4.4). We want to prove that the perpendicular bisector of AB passes through O. In other words, we wish to prove that $OC \perp AB$. In the triangles AOC and BOC we have $OA = OB$ (radius of the circle), $AC = CB$ (by hypothesis) and OC is common. Therefore the two triangles are congruent. Hence $\Delta OCA \equiv \Delta OCB$ which implies that $\angle OCA = \angle OCB = 90°$ or $OC \perp AB$.

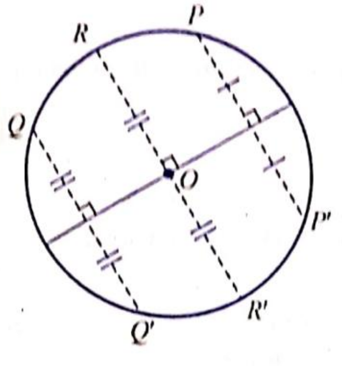


Fig. 4.2

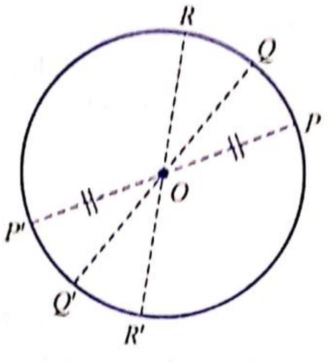


Fig. 4.3

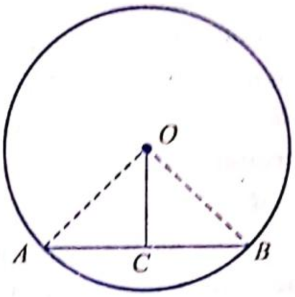


Fig. 4.4

Corollary. A circle is symmetrical about any of its diameters.

Proof. Let AB be any diameter of a circle with centre O (Fig. 4.5). If P is any point on the circle draw $PQ \perp AB$ meeting the circle again at Q. Then by Theorem 1, $PN = NQ$ (Fig. 4.5). Thus the circle is symmetrical about AB. ❑

Theorem 2. Given any three non-collinear points A, B, C there exists a unique circle passing through A, B and C.

Proof. Let A, B, C be any three non-collinear points. Suppose the perpendicular bisectors of BC and CA meet at S (Fig. 4.6). Then S lies on the perpendicular bisector of BC implies that $SB = SC$ (Theorem 12, Chapter 3). Again, S is also on the perpendicular bisector of CA implies that $SC = SA$. Hence we have $SA = SB = SC$. Further we note that any point equidistant from A, B, and C should lie on the perpendicular bisectors of BC and CA. Therefore, S is the only point equidistant from A, B, and C and so the circle with centre S and radius SA is the unique circle passing through A, B and C.

❑

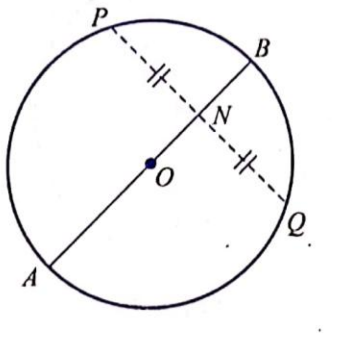


Fig. 4.5

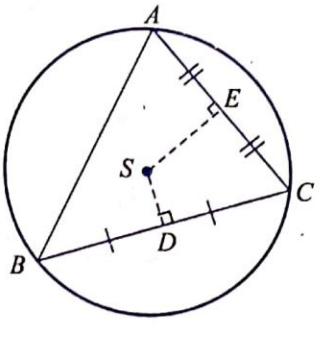


Fig. 4.6

We saw in the previous chapter that the perpendicular bisectors of the sides of a triangle concur at a point. We observe that the point of concurrence is the centre of the unique circle passing through the vertices of the triangle; hence it is called the circumcentre of the triangle and the corresponding circle is called the circumcircle of the triangle.

Corollary 1. If two circles have three points in common, then they must coincide.

Proof. Immediate from the theorem. ❑

Corollary 2. Two circles cannot intersect in more than two points.

Proof. Immediate from Corollary 1. ❑

Corollary 3. If A, B, C are any three points on a circle and O is a point within the circle such that $OA = OB = OC$ then O is the centre of the circle. ❑

Proof. It is immediate from the theorem.

Corollary 4. Two circles cannot have a common arc unless they coincide.

Proof. Any arc of a circle contains infinitely many points. Hence by Corollary 1, if two circles have a common arc, then they coincide. ❑

Theorem 3. Equal chords of a circle are equidistant from the centre. Conversely, if two chords of a circle are equidistant from the centre, then they are equal.

Proof. Suppose AB and CD are two equal chords of a circle with centre O. Let OX, OY be the perpendiculars from O onto the chords AB, CD respectively (Fig. 4.7). It is required to prove that $OX = OY$. By Theorem 1, OX and OY are the perpendicular bisectors of AB and CD. Hence $AX = CY = 1/2\, AB = 1/2\, CD$. Now, in ΔAOX and ΔCOY we have $\angle AXO = \angle CYO = 90°$, hypotenuse AO = hypotenuse CO (radii) and $AX = CY$. Therefore $\Delta AOX \equiv \Delta COY$ and hence $OX = OY$. Thus, the two equal chords AB and CD are equidistant from the centre O.

Converse. We may use the same figure, Fig. 4.7. But now we assume $OX = OY$ and we wish to prove that $AB = CD$. Again, we compare the right triangles AOX and COY. We have hypotenuse AO = hypotenuse CO and $OX = OY$. Therefore, the two triangles are congruent and hence we get $AX = CY$. But by Theorem 1, $AX = (1/2)\, AB$ and $CY = (1/2)\, CD$. Therefore, $AX = CY$ implies that $AB = CD$. ❑

Theorem 4. Given any two chords of a circle, the one which is nearer to the centre is greater than the one more remote.

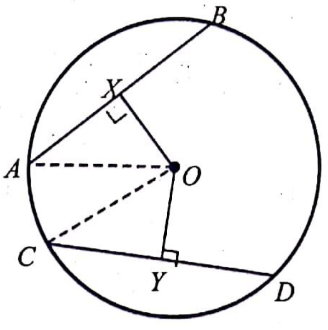


Fig. 4.7

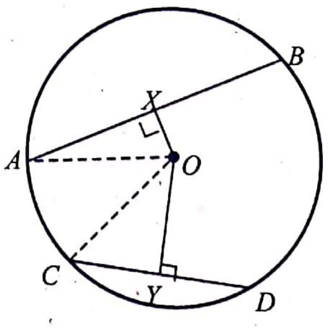


Fig. 4.8

Proof. Let AB, CD be two chords of a circle with centre O. Let OX, OY be the perpendiculars to AB, CD meeting them at X, Y respectively. (Fig. 4.8). Suppose $OX < OY$. Then from the right triangles AOX and COY we have $AO^2 = OX^2 + AX^2$ and $CO^2 = OY^2 + CY^2$. But $AO = CO$ = radius of the circle. Therefore we get $OX^2 + AX^2 = OY^2 + CY^2$. By assumption $OX < OY$ and so $OX^2 + AX^2 = OY^2 + CY^2$ can hold good if and only if $AX > CY$. But by Theorem 1, $AX = (1/2)\, AB$ and $CY = (1/2)\, CD$. ❑

Thus $OX < OY$ implies that $AB > CD$.

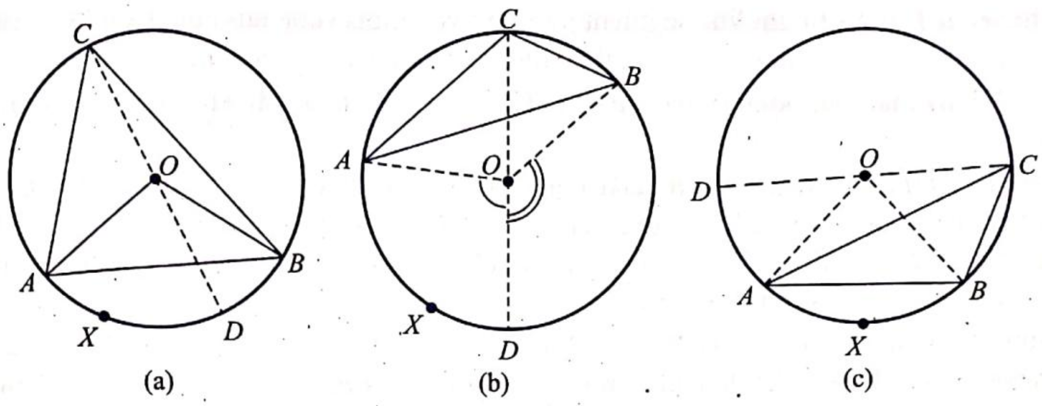


Fig. 4.9

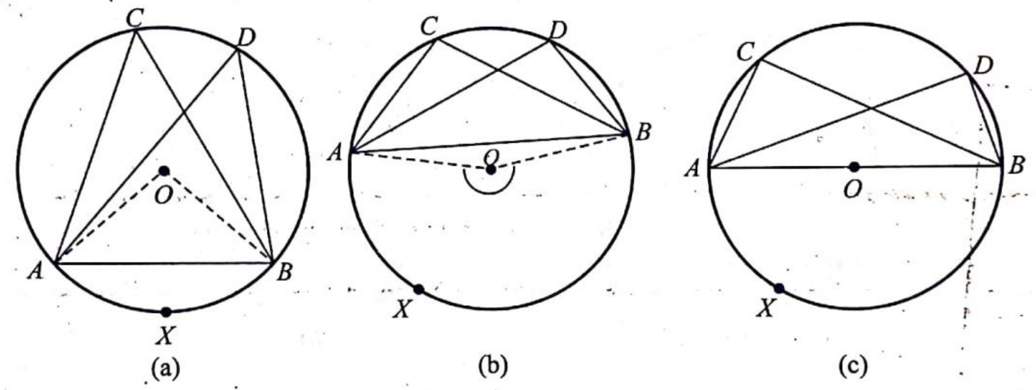


Fig. 4.10

Theorem 5. The angle subtended at the centre is double the angle subtended at any point on the remaining part of the circumference, for any arc of a circle.

Proof. Let AXB be an arc of a circle with centre O and C be any point on the remaining part of the circumference. It is required to prove that $\angle AOB = 2\ \angle ACB$. Let CO meet the circle again at D. Suppose O lies within the angle ACB as in Fig. 4.9(a) or Fig. 4.9(b). The triangles AOC and BOC are isosceles (Why?) and so we have $\angle OAC = \angle OCA$ and $\angle OBC = \angle OCB$. This gives, $\angle AOD = 2\ \angle ACO$ and $\angle BOD = 2\ \angle BCO$. Adding, we get $\angle AOD + \angle BOD = \angle AOB = 2(\angle ACO + \angle BCO) = 2\ \angle ACB$. If O is not within $\angle ACB$ as in Fig. 4.9(c), then we have $\angle BOD - \angle AOD = 2(\angle BCO - \angle ACO)$ which in turn gives $\angle AOB = 2\angle ACB$. ❑

Theorem 6. Angles in the same segment of a circle are equal.

Proof. Let ACB, ADB be two angles in the same segment $ACDB$ of a circle with centre O.

Then by Theorem 5, we see that $\angle ACB, = (1/2)\ \angle AOB = \angle ADB$. Hence, angles in the same segment are equal. (See Fig. 4.10(a) and 4.10(b)). ❑

If the segment happens to be a semicircle as in Fig. 4.10(c), then $\angle AOB = 2$ right angles $= 180°$ sines AOB is a straight angle in this case. This observation leads to the following corollary.

Corollary. The angle in a semicircle is always 90° ❑

Theorem 7. If a straight line segment joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

[We say that four points are *concyclic* if there is a circle passing through all the four points].

Proof. Let the line segment AB make equal angles at C and D, *i.e.*, $\angle ACB = \angle ADB$ as in Fig. 4.11. Then we wish to prove that A, B, C, D are concyclic. Draw the circle ABC passing through A, B and C. Suppose this circle does not pass through D; let it cut AD (or AD produced as in Fig. 11) at E.

Now, by construction, $\angle ACB = \angle AEB$ (angles in the same segment). But by our hypothesis $\angle ACB = \angle ADB$ and therefore $\angle ADB = \angle AEB$. This means that the exterior angle AEB of ΔEDB is equal to an interior opposite angle of the same triangle, which is impossible. Hence the circle ABC passes through D or in other words A, B, C, D are concylic. ❑

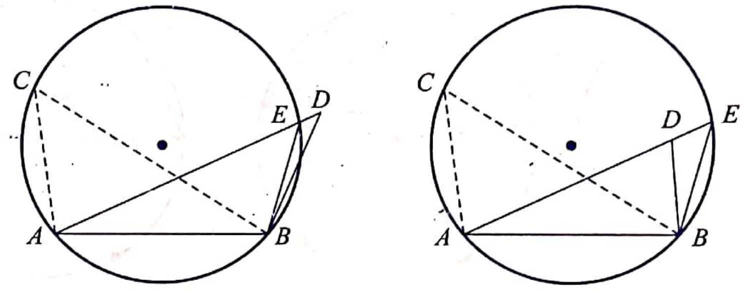


Fig. 4.11

EXERCISE 4.1

1. OA, OB are two equal line segments; the circle with centre C and radius r passes through A, B. Prove that $\angle AOC = \angle BOC$.
2. If A, B are any points on the circle S, prove that the chord AB which joins them lies entirely within the circle.
3. When two circles (C_1, r_1) and (C_2, r_2) cut, prove that $r_1 - r_2 < d < r_1 + r_2$ where $d = C_1C_2$.
4. When two circles cut each other then prove that the line joining their centres bisects their common chord at right angles.
5. A chord PQ of a circle cuts a concentric circle at P', Q'. Prove that $PP' = QQ'$.
6. Prove that two chords AB, CD of a circle bisect each other if and only if both of them are diameters.
7. Two circles cut each other at A, B. If PAQ and RBS are parallel straight lines meeting the circles again at P, Q, R, S, then prove that $PQ = RS$.
8. Show that the locus of the midpoints of a family of parallel chords of a circle is a diameter which is perpendicular to the given family of chords.
9. If $PQRS$ is a parallelogram whose vertices lie on a circle then PR and QS are diameters of the circle (see problem 6).
10. Prove that every circle passing through a fixed point and having its centre on a fixed straight line must pass though another fixed point.
11. Two straight lines OAB and OCD are drawn from an external point O to cut a given circle at A, B, C, D. Prove that the intersection of AD and BC cannot be the centre of the circle.

12. $ABCD$ is an isosceles trapezium. Prove that a circle can be drawn passing through A, B, C and D.
13. C is the midpoint of an arc ACB of a circle. Prove that C is equidistant from the radius through A and B.
14. AB and CD are two diameters of a circle and CE is a chord parallel to AB. Prove that B is the midpoint of the arc DBE.
15. $ABCD$ is a quadrilateral inscribed in a circle such that $AB = CD$. Prove that $AC = BD$.
16. A, B, C are three points on a circle and D, E are the midpoints of the minor arcs cut off by AB, AC. Prove that DE is equally inclined to AB and AC.
17. O is the centre of the circumcircle of an acute angled ΔABC. Show that $\angle OBC$ is the complement of $\angle BAC$.
18. ABC and $A'B'C'$ are two triangles such that $\angle A = \angle A'$ and $BC = B'C'$. Prove that the circumcircle of ΔABC is equal to the circumcircle of triangle $A'B'C'$.
19. $ABCD$ is a quadrilateral inscribed in a circle. If the diagonals AC and BD are at right angles, show that AB and CD subtend supplementary angles at the centre.
20. Find locus of middle points of chords of a circle which pass through a fixed point.
21. Given the base and the vertical angle of a triangle show that its area is greatest when it is isoscel.
22. ABC is a triangle and A', B', C' are the midpoints of the sides BC, CA, AB respectively. If AD is the altitude though A, prove that $\angle BDC = \angle BCA$. Hence show that the circumcircle of $A'B'C$ also passes through the feet D, E, F of the altitudes of triangle ABC.
23. Two circles intersect at A and B; PAQ is a straight line through A meeting die circles again at P, Q. Find the locus of the midpoint of PQ.
24. The circumferences of three unequal circles whose centres are A, B and C pass through a common point, O, from which lines are drawn through A, B and C meeting the circumferences at A', B' and C'. Show that the sides of $\Delta A'B'C'$ pass through the other points of intersection of the circles and are respectively parallel to the sides of the ΔABC.
25. A chord of constant length slides round a fixed circle. Show that the locus of any point fixed in the chord is a concentric circle.
26. A, B are the midpoints of two equal chords in a circle and the straight line joining A, B meets the circle at P, Q. Prove that $PA = QB$.
27. Prove that of all chords of a circle which are bisected by a fixed chord, the greater is that which meets the fixed chord at a point nearer its midpoint.
28. The internal bisector of $\angle A$ of ΔABC meets the circumcircle in D. If DE, DF are the perpendiculars to AB, AC respectively from D, then prove that $AE = (AB + AC)/2$.
29. If I is the incentre of ΔABC and AI meets the circumcircle at D, then prove that $DB = DC = DI$.
30. If H is the orthocentre of ΔABC and AH meets BC at D and the circumcircle at E, then prove that $HD = DE$.

4.2 TANGENTS

In general, we have seen that a straight line cuts a circle at, utmost two points. If a straight line has just one common point with a circle, we say that the straight line touches the circle. In that case, the straight line is called a *tangent* to the circle and the point at which a tangent touches the circle is known as the point of contact of the tangent. Two circles touch one another when they have only one point in common and have a common tangent at this point. Circles may touch externally in which case they

are on opposite sides of the common tangent; or they may touch internally, in which case they are on the same side of the common tangent (Fig. 4.12).

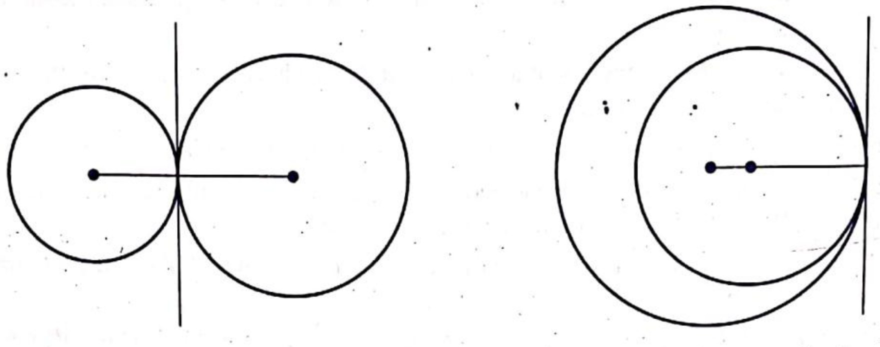


Fig. 4.12

Theorem 8. One and only one tangent can be drawn to a circle at any point on its circumference and this tangent is perpendicular to the radius through the point of contact.

Proof. Let P be any point on a circle with centre O.

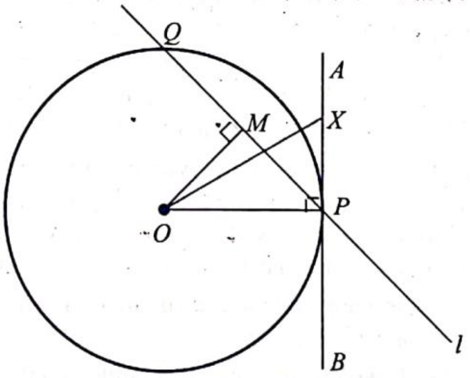


Fig. 4.13

Draw $APB \perp OP$ as in Fig. 4.13. If X is any point on the straight line APB different from P, then $\angle OPX$ is a right triangle with OX as its hypotenuse. Therefore $OX > OP$ = the radius of the circle. This means that X lies outside the given circle. This is true for every point X on the straight line APB except P. Hence the straight line AB touches the circle at P or in other words, the straight line through P perpendicular to the radius OP is a tangent to the circle at P. If any other straight line l through P is considered, let M be the foot of the perpendicular from O on l. Then as l is not perpendicular to OP, we see that $M \neq P$. On this straight line cut off MQ equal to PM (Fig. 4.13). Then by construction, OM is the perpendicular bisector of PQ; and therefore $OP = OQ$. This means that the point Q also lies on the given circle; and the straight line l cuts the circle at two distinct points P and Q. This says that l is not a tangent to the circle. Hence the theorem. ❑

Theorem 9. If two tangents are drawn to a circle from an exterior point then (*i*) the lengths of the tangents are equal (*ii*) they subtend equal angles at the centre (*iii*) the angle between them is bisected by the straight line joining the point and the centre.

Proof. See Fig. 4.14. Let A be an exterior point to the circle with centre O and AP, AQ be two tangents from A to the circle touching the circle at P and Q respectively. Then it is required to prove that (*i*) $AP = AQ$ (*ii*) $\angle AOP = \angle AOQ$ (*iii*) $\angle PAO = \angle QAO$. As AP and AQ are the tangents to the circle at P and Q. We have $\angle APO = \angle AQO = 90°$. We note that the right angled triangles OAP and OAQ are congruent (RHS theorem). Therefore $AP = AQ$, $\angle AOP = \angle AOQ$ and $\angle PAO = \angle QAO$. ❑

Given a circle and a point A exterior to it, how many tangents to the circle can be drawn through A? The following theorem answers this question.

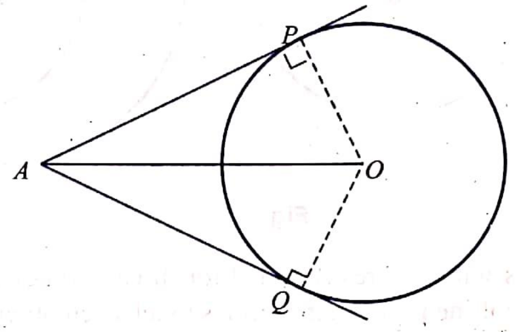


Fig. 4.14

Theorem 10. There are exactly two tangents from an exterior point to a given circle.

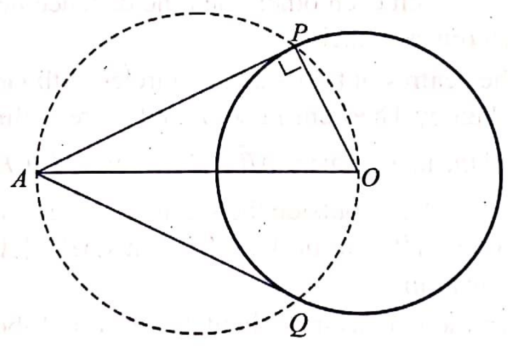


Fig. 4.15

Proof. Suppose P is the point of contact of a tangent to the circle from A. Then as $\angle APO = 90°$, P must lie on the circle AO as diameter (Theorem 7). Now, the circle on AO as diameter and the given circle cut exactly at two distinct points since A lies outside the given circle. (Fig. 4.15). Therefore, the points of contact of the tangents from A to the given circle must be the two points of intersection of the circle on AO as diameter and the given circle. Thus there are exactly two tangents from an exterior point to a given circle. ❑

We have already seen that if A lies on the circle, there is a unique tangent to the circle through A. If A lies inside the circle and AP is a tangent to the circle with P as its point of contact, then ΔAPO must be a right angled triangle, right angled at P. Therefore $AO^2 = AP^2 + OP^2$ and $AP^2 = AO^2 - OP^2$. But A lies inside the circle implies that $AO^2 - OP^2 < 0$. This gives $AP^2 < 0$ which is impossible, since the square of any real number is always non-negative. Hence there is no tangent to the circle through an interior point of the circle.

Theorem 11. If two circles touch one another, then the point of contact lies on the straight line joining the centres.

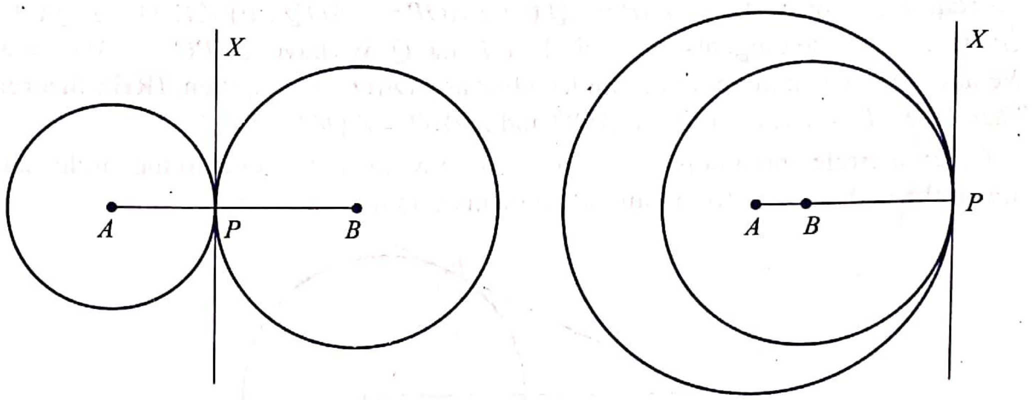


Fig. 4.16

Proof. Let two circles with centres A and B touch each other at P. It is required to prove that A, P, B are collinear. Since the circles touch each other at P (Fig. 4.16), they have a common tangent PX at P. Hence PA and PB are both perpendicular to the common tangent XP at P. This is possible only if A, P and B are collinear and the straight line AB is perpendicular to the common tangent. ❑

Corollary. If two circles touch each other, then the distance between their centres is equal to the sum or difference of their radii.

Proof. If A and B are the centres of two touching circles with radii r_1 and r_2 and if P is their point of contact, then by Theorem 11. A, P and B are collinear. When the circles touch externally P lies in the line segment $\overline{AB}$ and we have $AP + PB = AB$ or $r_1 + r_2 = AB$. When they touch internally, P lies outside the segment $\overline{AB}$ and we have $AB = AP - BP$ or $BP - AP$ depending upon $AP \geq BP$ or $AP \leq BP$. Thus $AB = |AP \pm BP| = |r_1 \pm r_2| =$ sum or difference of their radii. ❑

Remark. The perfect symmetry of a circle about its centre and about any of its diameters tells us that equal arcs subtend equal angles at the centre; and conversely if two arcs subtend equal angles at the centre then they are equal in length.

Theorem 12. In equal circles (or in the same circle) if two chords are equal, then they cut off equal arcs on the circles.

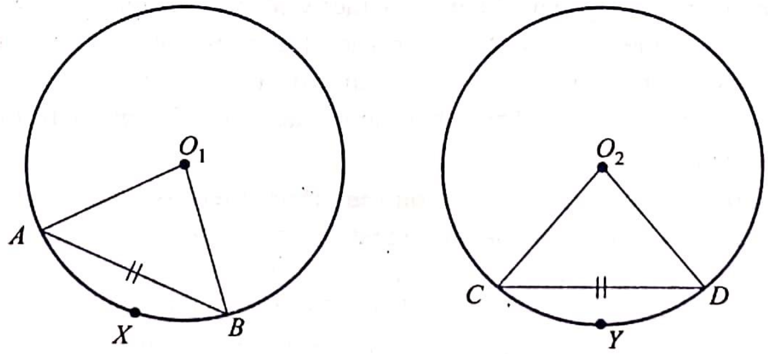


Fig. 4.17

Proof. Let AB and CD be two equal chords of two equal circles with centres O_1 and O_2 respectively. It is required to prove that arc AXB = arc CYD in length. By the *SSS* theorem $\Delta AO_1B = \Delta CO_2D$ and so $\angle AO_1B = \Delta CO_2D$. Therefore by our remark preceding the theorem arc AXB = arc CXD in length. ❑

The converse of the above theorem, namely, "In equal circles (or in the same circle) if two arcs are equal then they cut off equal chords on the circles" is also true. The proof is left as an exercise

Theorem 13. In any circle, the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle in the alternate segment.

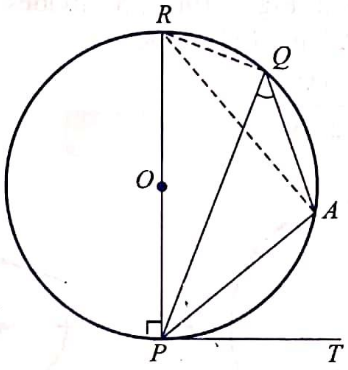


Fig. 4.18

Proof. Let PT be a tangent to a circle with centre O, the point of contact being P. Let AP be any chord through P. It is required to prove that $\angle APT = \angle AQP$ where Q is any point on the other segment determined by AP. Let PR be the diameter through P. Then $\angle PAR = 90°$ (angle in a semi-circle) and $\angle RPT = 90°$ (Fig. 4.18). Therefore $\angle ARP = 90° - \angle APR$ (from the right triangle RAP) $= \angle TPR - \angle APR = \angle APT$. But $\angle ARP = \angle AQP$ and hence $\angle APT = \angle AQP$ = angle in the alternate segment. ❑

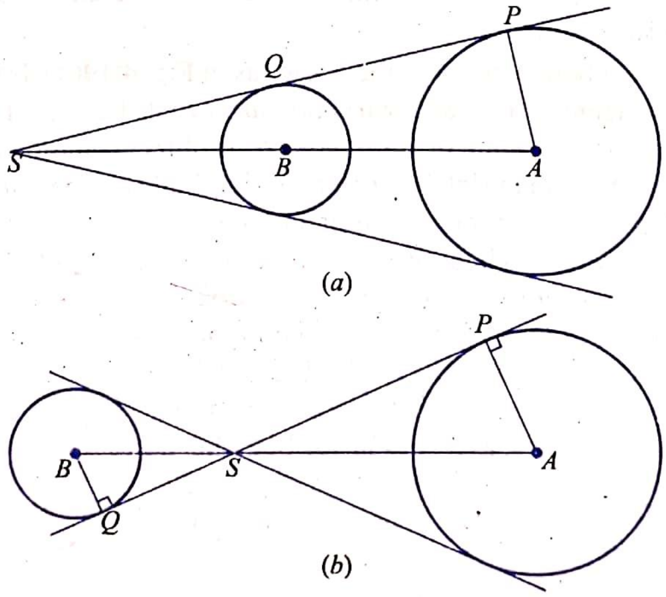


Fig. 4.19

Theorem 14. A common tangent to two circles divides the straight line segment joining their centres, externally or internally in the ratio of their radii.

Proof. Suppose PQ is a common tangent to the two circles with centres A, B and radii r_1, r_2 respectively such that P and Q are the points of contact with the corresponding circles. Let PQ meet the line of centres joining A and B at S (Fig. 4.19(a)) or S' (Fig. 4.19(b)). We have $\angle APS = \angle BQS = 90°$. Therefore $AP \parallel BQ$ and the triangles BQS and APS are equiangular and hence similar.

So, $\dfrac{AS}{SB} = \dfrac{AP}{BQ} = \dfrac{r_1}{r_2}$ and hence S divides AB externally in the ratio $r_1 : r_2$.

When PQ meets AB at S' as in Fig. 4.19(b), S' divides AB internally and again the similarity of the triangles APS' and BQS' gives

$$\frac{AS'}{S'B} = \frac{r_1}{r_2}$$

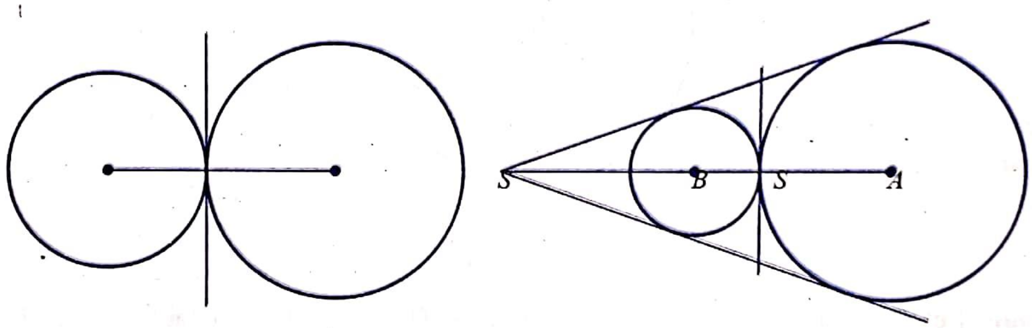


Fig. 4.20

Thus any common tangent to two circles divides the straight line segment joining their centres either internally or externally in the ratio of the radii. □

Definition 1. The points S and S' dividing the line segment joining the centres of two circles internally and externally in the ratio of their radii are known as the *centres of similitude* of the two circles.

When S and S' are both exterior to the circles as in Fig. 4.19(a), Fig. 4.19(b), there are two common tangents from S and two common tangents from S'. The two common tangents from the external centre of similitude are the direct common tangents and the two common tangents from the internal centre of similitude are the transverse common tangents. Thus in general, there are four common tangents to two circles. When two circles touch externally (Fig. 4.20) there is only one transverse common tangent and there are two direct common tangents. When two circles touch internally (Fig. 4.21) there is only one direct common tangent and no transverse common tangents, as S' lies inside both the circles. Also when two circles cut each other, there are two direct common tangents and no transverse common tangents (Fig. 4.22). When one circle lies entirely within the other there are no common tangents.

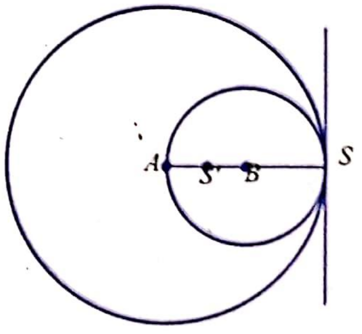


Fig. 4.21

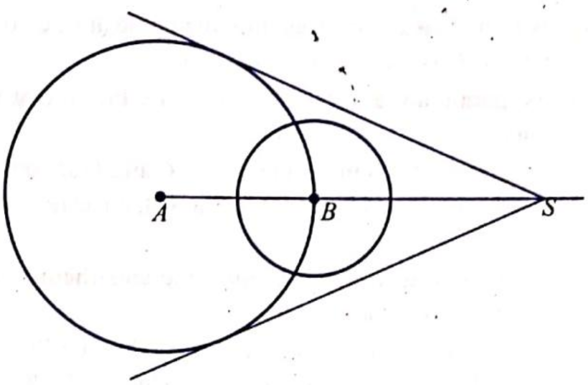


Fig. 4.22

EXERCISE 4.2

1. If PA, PB are tangents to a circle whose centre is O, then prove that $\angle APB + \angle AOB = 180°$.
2. If P, Q and R, S are the points of contact of the two direct common tangents to two circles prove that $PQ = RS$.
3. If P, Q and R, S are the points of contact of the two transverse common tangents to two circles prove that $PQ = RS$.
4. If two circles intersect at A, B prove that the angle between the tangents at A is the same as the angle between the tangents at B.
5. If a circle can be inscribed in a quadrilateral, prove that the sum of one pair of opposite sides is equal to the sum of the other pair.
6. If a circle can be inscribed in a parallelogram, prove that the parallelogram is a rhombus.
7. If a straight line cuts a circle at A, B prove that it cuts the circle at the same angle at each of these points.
8. What is the locus of the centre of a circle which touches two given parallel straight lines?
9. S_1 and S_2 are two concentric circles and AB is a chord of outer circle S_1 touching S_2 at C. Prove that $AC = CB$.
10. What is the locus of the centres of circles which touch a given circle at a given point?
11. What is the locus of centres of circles of given radius which touch a given circle?
12. Two circles with centres A, B touch at P. If XPY is drawn to cut the circles again at X, Y, prove that $AX \parallel BY$.
13. Two circles intersect in A and a straight line XAY is drawn to cut the circles again at X and Y. Tangents at X and Y to the respective circles cut at Z. Prove that $\angle XZY$ is equal to the angle between the tangents at A.
14. A straight line cuts two concentric circles in A_1, A_2 and B_1, B_2. Prove that the four intersections of a tangent at an 'A' and at a 'B' lie on another concentric circle.
15. A straight line of given length subtends given equal angles at two fixed points. Prove that the straight line a always touches a fixed circle.
16. Three equal circles pass through a given point H and meet one another two by two at A, B, C. Prove that H is the orthocentre of ΔABC.
17. A tangent to a circle at a point P on it, is parallel to the chord AB. Prove that P bisects the arc cut off by AB.
18. AB is a chord of a circle and AT is the tangent at A. Prove that the bisector $\angle BAT$ bisects the arc AB.

19. AB, AC are tangents from A to a circle touching the circle at B, C. If D is the midpoint of minor arc BC, prove that D is the incentre of ΔABC.

20. The diagonals of the parallelogram $ABCD$ meet at O. Prove that the circles AOB and COD touch each other.

21. AB is a chord of a circle and PAQ is the tangent at A; C and D are points on the circle such that CA and DA bisect the angles BAP and BAQ. Show that CD is a diameter perpendicular to AB.

22. Two circles touch internally at X and a straight line cuts them at A, B, C, D in order. Prove that AB, CD subtend equal angles at X.

23. Suppose the internal and external bisector of $\angle A$ meet the side BC and BC (produced) at E and F respectively. If the tangent at A to the circle ABC meets BC produced at D, prove that D bisects EF.

24. A triangle ABC circumscribes a circle, with points of contact being X, Y, Z. If the feet of the altitudes of ΔXYZ are D, E, F prove that the sides of ΔDEF are parallel to the sides of ΔABC.

If in a quadrilateral, the sum of one pair of opposite sides is equal to the sum of the other pair, prove that a circle can be inscribed in it. (See problem 4- Problems Ch. 3)

26. If a circle can be inscribed in a quadrilateral prove that the bisectors of the angles of the quadrilateral are concurrent.

27. Four circles of different radii are such that each circle touches two and only two of the others. Show that a circle can be inscribed in the quadrilateral $ABCD$ where A, B, C, D are the centres of the given four circles.

28. If the circumference of a circle is divided into n equal parts, prove that

(*i*) the points of division are the vertices of a regular polygon.

(*ii*) the tangents at the points of division are the sides of a regular polygon.

29. Two circles cut at right angles, (*i.e*, the angle between the tangents at their points of intersection is 90°). Show that the area common to the two circles together with the square on the radius is equal to the area of either circle.

30. Two given circles intersect at A and B. A straight line through B meets the circles again at C and D. (*i*) Show that CD is greatest when it is parallel to the line joining the centres. (*ii*) When is the area of ΔACD the greatest possible?

31. AB is a diameter of a circle and BM is the tangent at B. If the tangent at a point C on the circle meets BM at X and if AC produced meets BM at Y, prove that $BX = XY$,

32. Prove that if a chord and a tangent are drawn from a point on a circle, the midpoint of the subtended arc is equidistant from them.

33. A and B are points on two concentric circles. Prove that the angle between the tangents at A and B is the same as the angle subtended by AB at the centre.

34. Two circles are said to cut each other orthogonally if the angle between their tangents at a common point is a right angle. Show that the locus of the centres of circles cutting a given circle orthogonally at a given point is a straight line.

35. Find the locus of centres of circles of given radius cutting a given circle orthogonally.

36. If AB is a common tangent to two circles, prove that the circle on AB as diameter cuts each of the circles orthogonally.

37. If two circles of radii r_1, r_2 cut orthogonally at A, B prove that $AB \cdot d = 2r_1r_2$ where d is the distance between their centres.

38. If H is the orthocentre of ΔABC, show that the circles on AH and BC as diameters cut orthogonally.

39. O is a fixed point; P is a variable point on a fixed circle S. If P' is on the line OP such that $OP'/OP = \lambda$ = a constant, find the locus of P'.

40. O is a fixed point; P is a variable point on a fixed circle with centre C. The line bisecting $\angle OCP$ meets OP at P'. Find the locus of P'.

41. A straight line OPQ is drawn through a centre of similitude O of two circles to cut them at P and Q. Prove that the tangents at P, Q are parallel.

42. Two circles cut at A and B and their common tangents meet at O. If AP, AQ are the tangents at A to the two circles, prove that OA bisects $\angle PAQ$.

43. Prove that if three circles are tangent to one another, the tangents at the points of contact are concurrent.

4.3 CYCLIC QUADRILATERALS

A quadrilateral $ABCD$ is *cyclic if* there is a circle passing through all the four vertices of the quadrilateral.

Theorem 15. The opposite angles of a cyclic quadrilateral are supplementary.

Proof. Let $ABCD$ be a cyclic quadrilateral inscribed in a circle with centre O. It is required to prove that $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$. Now, by Theorem 5, $\alpha = 2\angle A$ and $\beta = 2\angle C$ (Fig. 4.23). Therefore, $360° = \alpha + \beta = 2(\angle A + \angle C)$ or $\angle A + \angle C = 180°$. Similarly $\angle B + \angle D = 180°$. ❑

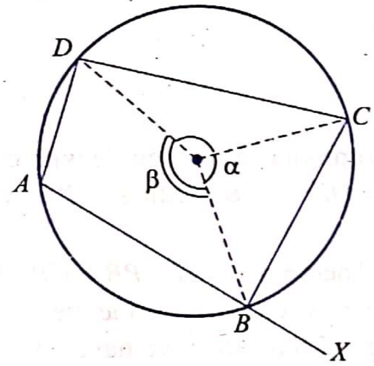


Fig. 4.23

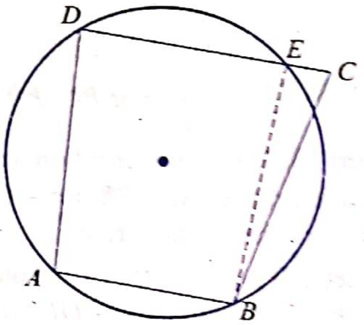


Fig. 4.24

Corollary. If $ABCD$ is a cyclic quadrilateral then any exterior angle of $ABCD$ is equal to the interior opposite angle.

Proof. We wish to prove that the exterior angle XBC is equal to the interior opposite angle ADC (Fig. 4.23); this is immediate since $\angle XBC = 180° - \angle B = \angle D$. ❑

Theorem 16. If two opposite angles of a quadrilateral are supplementary then it is cyclic.

Proof. Let $ABCD$ be a quadrilateral such that $\angle A + \angle C = \angle B + \angle D = 180°$. (See Fig. 4.24). Suppose the circle through A, B and D cuts the straight line DC at E. Then $ABED$ is a cyclic quadrilateral. Therefore $\angle BED = 180° - \angle BAD$. But $\angle BAD + \angle BCD = 180°$ by our hypothesis. Hence $\angle BED = \angle BCD$. If $C \neq E$, this says that an exterior angle of ΔBCE is equal to an interior opposite angle, which is impossible. Therefore $C = E$ or the quadrilateral $ABCD$ is cyclic. ❑

Note. The above proof also works when E lies on DC produced, in which case BCD is an exterior angle of ΔBCA.

Theorem 17. If AB and CD are any two chords of a circle meeting at a point P then $PA \cdot PB = PC \cdot PD$ (known as the secant property of a circle).

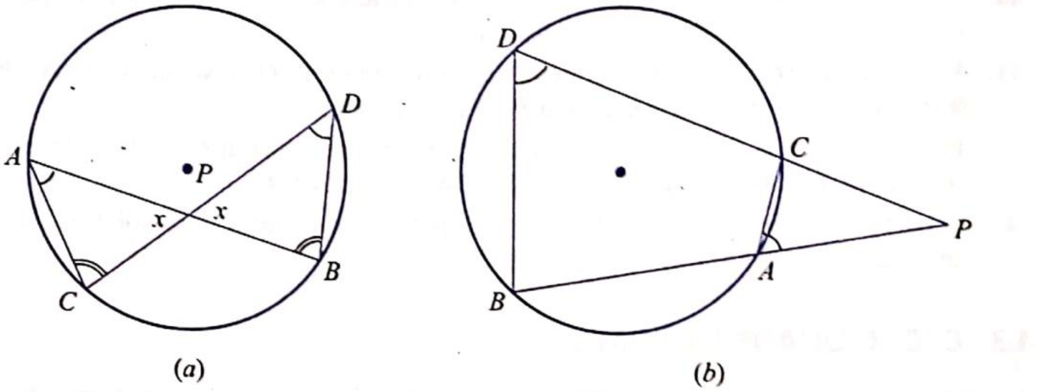


Fig. 4.25

Proof. In ΔAPC and ΔDPB we have

$\angle APC = \angle DPB$	(See Fig. 4.25(a) and Fig. 4.25(b))
$\angle PDB = \angle PAC$	(In Fig. 4.25(a) these are angles in the same segment and in Fig. 4.25(b) ext $\angle PAC$ = int. opp $\angle PDB$)
$\angle DBP = \angle ACP$	(reasons same as above).

Hence the two triangles are similar. This gives

$$\frac{PA}{PD} = \frac{PC}{PB} = \text{or } PA \cdot PB = PC \cdot PD.$$ ❑

Theorem 18. If P is any point on a chord AB (or AB produced) of a circle with centre O and radius r, then $AP \cdot PB = r^2 - OP^2$ or $PA \cdot PB = OP^2 - r^2$ according as P is within the circle or outside the circle.

Proof. Let CD be the diameter through P. Then by Theorem 17, $AP \cdot PB = CP \cdot PD = (CO - OP)\,(DO + OP) = r^2 - OP^2$ since $CO = DO = r$, when P lies inside the circle as in Fig. 4.26(a). If P lies outside the circle as in Fig. 4.26(b), then we have $PA \cdot PB = PC \cdot PD = (OP - OC)\,(OP + OD) = OP^2 - r^2$. ❑

Corollary. If P is any point on a chord AB produced of a circle with centre O and radius r then $PA \cdot PB = PT^2$ = (length of the tangent from $P)^2$.

Proof. Let PT be the tangent to the circle touching the circle a T (Fig. 4.27). Then PTO is a right triangle. $PT^2 = OP^2 - r^2 = PA \cdot PB$ (by Theorem 18). ❑

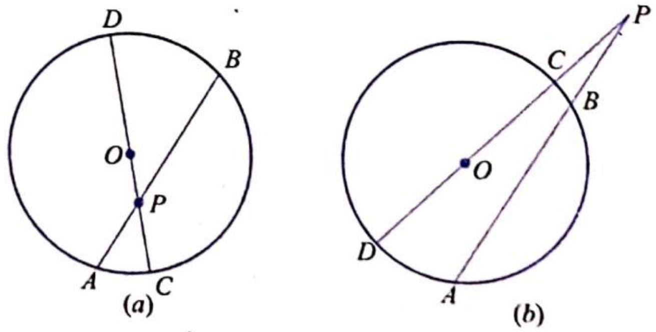


Fig. 4.26

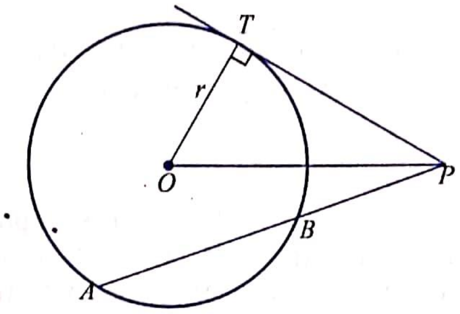


Fig. 4.27

Definition 2. If P is any point in the plane of a circle with centre O and radius r, the *power* of P with respect to the circle is defined as $OP^2 - r^2$. Thus if directed segments are used then $PA \cdot PB$ = Power of P with respect to the circle whenever P is a point on the chord AB (or AB produced).

We note that if P lies on a circle Σ, then the power of P with respect to Σ is zero; if P lies outside the circle, then the power of P is the square of the length of the tangent from P; and if P lies inside the circle, the power of P is negative.

Theorem 19. If two straight line segments AB and CD (or both being produced) intersect at P so that $PA \cdot PB = PC \cdot PD$ then the four points A, B, C, D are concyclic.

Proof. Let the circle through A, C and D cut AB or AB produced at E. (Fig. 4.28). Then by the secant property of a circle $PA \cdot PE = PC \cdot PD$. But by our hypothesis $PA \cdot PB = PC \cdot PD$ and hence $PE - PB$, which means that E coincides with B. Thus A, B, C, D are concyclic. ❑

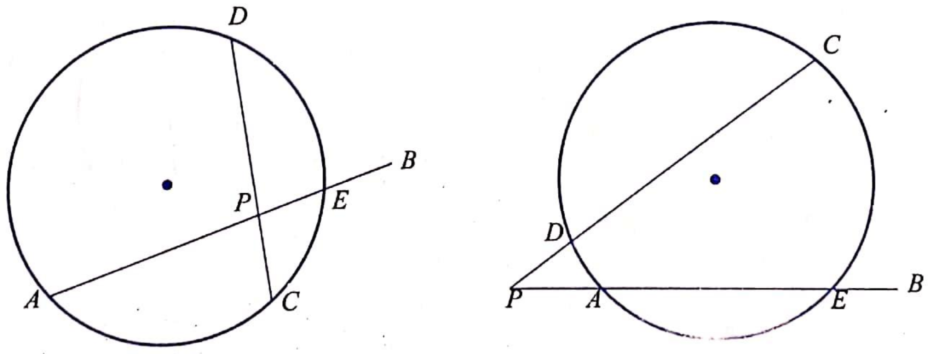


Fig. 4.28

Theorem 20. If AD bisects the vertical angle A of ΔABC meeting the base BC at D then $AB \cdot AC = BD \cdot DC + AD^2$.

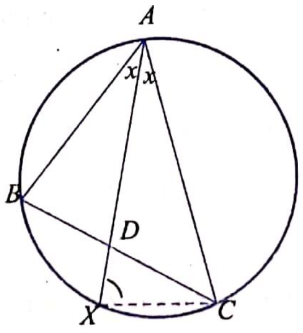


Fig. 4.29

Proof. Let AD cut the circumcircle of ΔABC again at X. Then $\angle ABD = \angle AXC$ being angles in the same segment determined by the chord AC. Also, $\angle BAD = \angle CAD$ as AD bisects $\angle A$. Therefore ΔABD ||| ΔAXC (Fig. 4.29) and

we get $$\frac{AB}{AX} = \frac{AD}{AC}.$$

This gives $AB \cdot AC = AD \cdot AX$. Also, $AD^2 = AD\,(AX - DX)$ implies that $AD \cdot AX = AD^2 + AD \cdot DX$. Therefore we get $AB \cdot AC = AD \cdot AX = AD^2 + AD \cdot DX$. By the secant property of a circle we have $AD \cdot DX = BD \cdot DC$. Hence $AB \cdot AC = BD \cdot DC + AD^2$. ❑

Theorem 21. If AD is the altitude through A of ΔABC and if R is the circumradius of ΔABC then $AB \cdot AC = 2R \cdot AD$.

Proof. Let AE be the diameter through A of the circle ABC. (Fig. 4.30). We have $\angle ADC = \angle ABE = 90°$ and $\angle ACD = \angle AEB$ (angles in the same segment).

Therefore ΔADC ||| ΔABE and so $\dfrac{AB}{AD} = \dfrac{AE}{AC}$.

This gives $AB \cdot AC = AE \cdot AD = 2R \cdot AD$. ❑

Corollary. $$\Delta = \text{Area of } ABC = \frac{abc}{4R} \text{ (usual notations)}$$

Proof. We have $$\Delta = \frac{1}{2} a \cdot AD = \frac{abc}{2.2R} = \frac{abc}{4R}$$ ❑

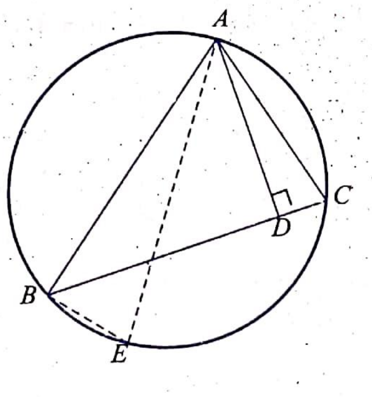


Fig. 4.30

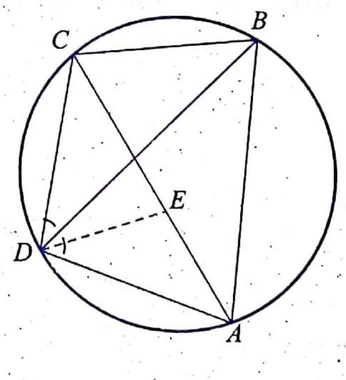


Fig. 4.31

Theorem 22. (Ptolemy's Theorem)

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides.

Proof. Let $ABCD$ be a cyclic quadrilateral. It is required to prove that $AC \cdot BD = AB \cdot CD + BC \cdot AD$. We may assume without loss of generality that $\angle ADB > \angle BDC$. Draw DE meeting AC at E such that $\angle ADE = \angle BDC$. (Fig. 4.31). Then the triangles ADE and BDC are equiangular and hence similar. Therefore

$$\frac{AD}{BD} = \frac{AE}{BC} \quad \text{or} \quad AD \cdot BC = BD \cdot AE \qquad (1)$$

Again, $\angle ADB = \angle ADE + \angle EDB = \angle BDC + \angle EDB = \angle EDC$

and $\angle DBA = \angle DCE$ (angles in the same segment).

Therefore $\Delta ADB \,|||\, \Delta EDC$ and so

$$\frac{AB}{EC} = \frac{BD}{CD} \text{ or } AB \cdot CD = BD \cdot EC \qquad (2)$$

Adding (1) and (2) we get $AB \cdot CD + BC \cdot AD = BD\,(AE + EC) = BD \cdot AC.$

Theorem 23. If $ABCD$ is a quadrilateral which is not cyclic then $AB \cdot CD + BC \cdot AD > AC \cdot BD$.

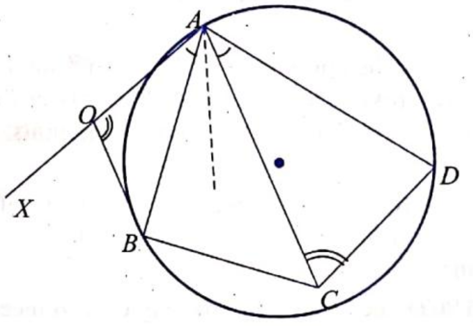


Fig. 4.32

Proof. Suppose $ABCD$ is not a cyclic quadrilateral, draw the circle ABD. Let AX be a straight line symmetric to AD about the bisector of $\angle BAC$. In other words, AX is a straight line such that $\angle XAB = \angle CAD$. There is a unique point O on AX such that $\angle AOB = \angle ACD$. Therefore, comparing the triangles AOB and ACD, the third pair of angles ABO and ADC must also be equal. This means that O does not lie on BC (otherwise $ABCD$ becomes a cyclic quadrilateral).

$\Delta AOB \,|||\, \Delta ACD$ gives $$\frac{AO}{AC} = \frac{AB}{AD} = \frac{OB}{CD} \qquad (1)$$

Also, $\Delta OAC \,|||\, \Delta BAD$ (Why?) gives $= \dfrac{OC}{BD} = \dfrac{AC}{AD}$. (2)

Hence $AC \cdot BD = OC \cdot AD < (OB + BC) \cdot AD = OB \cdot AD + BC \cdot AD$

$= AB \cdot CD + BC \cdot AD$. from (1) ❑

Note. With notations as in the proof of Theorem 23, we note that if $ABCD$ is cyclic, we must have O lying on BC and $OC = OB + BC$.

$AC \cdot BD = OC \cdot AD = (OB + BC) \cdot AD = OB \cdot AD + BC \cdot AD = AB \cdot CD + BC \cdot AD$ which is Ptolemy's theorem.

Corollary. Quadrilateral $ABCD$ is cyclic iff $AC \cdot BD = AB \cdot CD + AD \cdot BC$. ❑

Proof. Immediate from the theorem.

EXERCISE 4.3

1. $ABCD$ is a cyclic quadrilateral and AB, CD are produced to meet at X. Prove that ΔXAD and ΔXCB are similar.
2. If I is the incentre of ΔABC and I_a is the excentre opposite to A, prove that $BICI_a$ is a cyclic quadrilateral.

3. In ΔABC, AD and BE are the altitudes through A and B. If H is the orthocentre, prove that the quadrilateral $DHEC$ is cyclic. Hence show that $\angle AHB = \pi - \angle C$ and that the circles AHB and ACB are equal circles.
4. O is the centre of a circle and AB is a diameter of the circle. If OD is perpendicular to AB and meets a chord AC (or AC produced) at D, then prove that the circle AOD is equal to the circle through O, D, B, C.
5. Let P be any point on the circumcircle of triangle ABC and let L, M, N be the feet of the perpendiculars from P on the sides BC, CA, AB respectively. Prove that L, M, N are collinear.
6. In ΔABC let A', B', C', be the midpoints of BC, CA, AB and let H be the orthocentre. If P is the midpoint of AH, prove that the circle $A' B' C$ passes through P. Hence prove that the midpoints of the sides of triangle ABC and the midpoints of the line segments joining the orthocentre to the vertices all lie on a circle.
7. Using problem 6, prove that the circle through the midpoints of the sides and the midpoints of the lines joining the orthocentre to the vertices also passes through the feet of the altitudes of the triangle.
8. In a quadrilateral $ABCD$, the bisectors of the angles A, B meet at E and those of B, C; C, D; D, A meet in F, G, H respectively. Prove that $EFGH$ is cyclic.
9. If the exterior angles of a quadrilateral are bisected by four straight lines, prove that these four straight lines form a cyclic quadrilateral.
10. X, Y, Z are any three points on the sides BC, CA, AB of triangle ABC. Prove that the three circles AYZ, BZY and CXY meet at a point.
11. A, B, C are three collinear points and P is a point not in the line AB; AFE, BFD and CED are perpendiculars to PA, PB, PC respectively. Prove that P, D, E, F are concyclic.
12. $ABCD$ is a cyclic quadrilateral; the sides AB and DC are produced to meet at X and AD, BC produced meet at Y. Show that the bisectors of the angles BXC, CYD are at right angles.
13. If X is any point on the internal bisector of $\angle A$, prove that
$$\Delta BAX/\Delta CAX = BA/AC.$$
14. Find a point X inside ΔABC such that
$$\Delta AXB : \Delta BXC : \Delta CXA = k : l : m$$
where k, l, m are given constants.
15. The diagonals AC, BD of a cyclic quadrilateral $ABCD$ meet at O. Prove that $AB \cdot BC/AD \cdot DC = BO/OD$.
16. S_1 and S_2 are two circles touching internally at O, with S_2 being the inner circle. A straight line cuts S_1 at A, D and S_2 at B, C. Prove that
$$AB : CD = (OA \cdot OB) : (OC \cdot OD).$$
17. X is any point on the circle through the four vertices of a cyclic quadrilateral $ABCD$. If x, y, z, w, u, t are the perpendicular distances of X from AB, BC, CD, DA, AC, BD respectively, prove that $xz = yw = ut$.
18. AB is a chord of a circle and the tangents at A, B meet at C. If P is any point on the circle and PL, PM, PN are the perpendiculars from P to AB, BC, CA prove that $PL^2 = PM \cdot PN$.
19. X is any point on AB and the median AD of triangle ABC meets XC at Y. Prove that $XY/YC = AX/XB$.
20. AB is a diameter of a circle and PQ is a chord perpendicular to AB meeting AB at X. If the tangent at P meets AB at Y, prove that $YQ/QX = YP/PX$.
21. A, B are fixed points; AP and BQ are parallel chords of a variable circle such that AP/BQ is a constant. Prove that the locus of P is a circle.

22. Two given circles subtend equal angles at a point P. Find the locus of P.

23. X is any point on the minor arc BC of the circum circle of an equilateral triangle ABC. Prove that $XA = XB + XC$.

24. ABC is an isosceles triangle with $AB = AC$. The altitude AD meets the circumcircle at P. Prove that $AP \cdot BC = 2AB \cdot BP$.

25. $ABCDE$ is a regular pentagon; P is any point on the minor arc AB of the circum circle of $ABCDE$. Prove that $PA + PB + PD = PC + PE$.

26. P is any point inside a parallelogram $ABCD$ such that $\angle APB + \angle CPD = 180°$. Prove that $AP \cdot CP + BP \cdot DP = AB \cdot BC$.

4.4 TRIANGLES REVISITED

Note. Here and in the rest of the book, the angles of a ΔABC may be denoted by A, B, C instead of $\angle A, \angle B, \angle C$ wherever the context is clear like for example, $A + B + C = 180°$.

Theorem 24. Let ABC be a triangle, AD the altitude through A and AK the circumdiameter through A. Then $\angle DAK = \angle B - \angle C$. Further the angular bisector AX of $\angle A$ bisects $\angle DAK$.

Proof. We have $\angle ABC = \angle AKC$ (angles in the same segment).

$\angle BAD = 90° - \angle ABC = 90° - \angle AKC = \angle KAC$ (Fig. 33)

$\angle DAK = \angle BAC - 2\angle BAD = \angle A - 2(90° - \angle B)$

$= \angle A + 2\angle B - 180° = \angle B - \angle C$ since $A + B + C = 180°$.

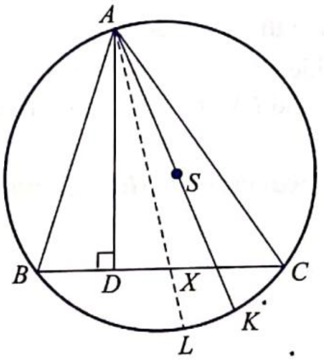


Fig. 4.33

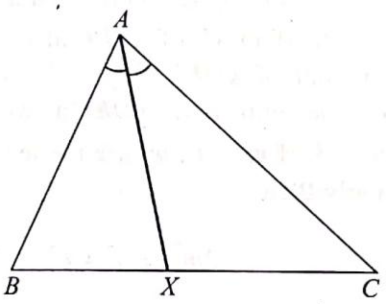


Fig. 4.34

This proves the first part of the theorem. We have taken B, C both acute in Fig. 4.33. The same proof works when one of B and C is obtuse. Let AXL be the angular bisector of $\angle A$ (Fig. 4.33). We have

$$DAX = \frac{\angle A}{2} - \angle BAD = \frac{\angle A}{2} - \angle CAK = \angle XAK. \qquad \square$$

Thus AX also bisects DAK.

Theorem 25. If the internal bisector of $\angle A$ of ΔABC meets BC at X then the difference between $\angle AXB$ and $\angle AXC$ is the same as the difference between $\angle B$ and $\angle C$.

Proof. We have $\angle AXC = \angle B + \dfrac{\angle A}{2}$ and $\angle AXB = \angle C + \dfrac{\angle A}{2}$.

Therefore $\angle AXB - \angle AXC = \angle B - \angle C$. $\square$

Theorem 26. In ΔABC, if the internal and external bisectors of $\angle A$ meet the circumcircle at X and Y, then XY is a circumdiameter perpendicular to BC.

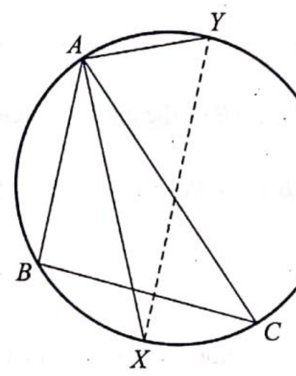


Fig. 4.35

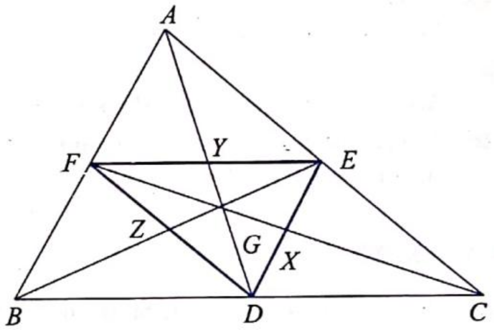


Fig. 4.36

Proof. Since equal arcs subtend equal angles at the circumference, $\angle BAX = \angle XAC$ gives arc BX = arc XC (Fig. 4.35). Therefore the diameter XY' of the circum circle should be the perpendicular bisector of the chord BC. Now, XY' is a diameter implies that $AY' \perp AX$. This means that AY' must be the external bisector of $\angle A$. Hence Y' coincides with Y and $XY \perp BC$. ❑

If ABC is a triangle with D, E, F as the midpoints of the sides BC, CA, AB respectively, then ΔDEF is called the *medial triangle* of ABC.

Theorem 27. A triangle and its medial triangle have the same centroid.

Proof. The median AD of ΔABC also bisects the side EF of ΔDEF (Why?). Therefore DY is a median of ΔDEF (Fig. 4.36). Similarly EZ and FX are also medians of ΔDEF. Thus G is the centroid of ΔDEF as well. ❑

Theorem 28. If m_a, m_b, m_c are the lengths of the medians of ΔABC, through A, B, C respectively then

$$2m_a^2 = b^2 + c^2 - \frac{a^2}{2},$$

$$2m_b^2 = c^2 + a^2 - \frac{b^2}{2} \quad \text{and} \quad 2m_c^2 = a^2 + b^2 - \frac{c^2}{2},$$

where a, b, c are the lengths of the sides BC, CA, AB of ΔABC.

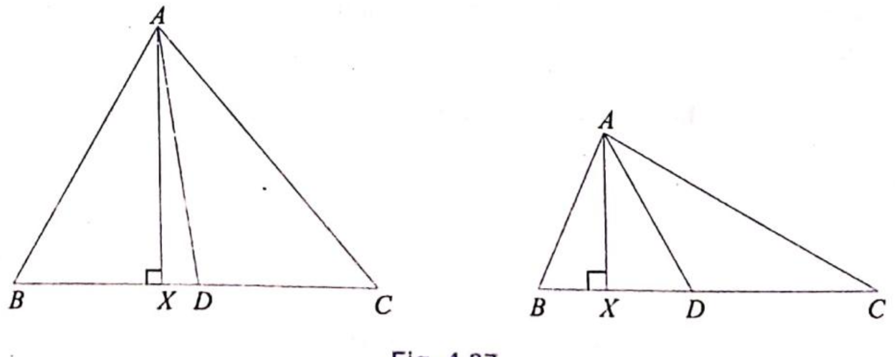


Fig. 4.37

Proof. Let AD be the median through A and AX the altitude through A. We use Pythagoras's theorem repeatedly.

We have $AB^2 = AX^2 + XB^2 = (AD^2 - DX^2) + XB^2$ (Fig. 4.37)

$= AD^2 + (DB - DX)^2 - DX^2$

$= AD^2 + DB^2 - 2DB \cdot DX.$

Similarly, $AC^2 = AD^2 + DC^2 + 2DC \cdot DX$

Adding we get $AB^2 + AC^2 = 2AD^2 + 2DB^2$ (since $DB = DC$)

or $$2AD^2 = AB^2 + AC^2 - \frac{1}{2}(BC^2) \text{ as } DB = \frac{BC}{2}$$

i.e., $$2m_a^2 = b^2 + c^2 - \frac{a^2}{2}.$$

Similarly $$2m_b^2 = c^2 + a^2 - \frac{b^2}{2} \quad \text{and} \quad 2m_c^2 = a^2 + b^2 - \frac{c^2}{2}.$$ ❑

Corollary 1. 1. $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(b^2 + c^2 + a^2)$

2. $GA^2 + GB^2 + GC^2 = \frac{1}{3}(b^2 + c^2 + a^2)$

where G is the centroid of ΔABC.

Proof. 1. Follows immediately from Theorem 28.

2. We have $GA = (2/3)\, m_a$, $GB = (2/3)\, m_b$ and $GC = (2/3)\, m_c$.

$$GA^2 + GB^2 + GC^2 = (4/9)\ (m_a^2 + m_b^2 + m_c^2) = (1/3)\ (a^2 + b^2 + c^2).$$ ❑

Corollary 2. If P is any point in the plane of ΔABC then

$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2$ where G is the centroid of ΔABC.

Proof. Let X be the midpoint of AG. (Fig. 4.38). The median PD of ΔPBC is given by

$$2PD^2 = PB^2 + PC^2 - \frac{BC^2}{2} \qquad (1)$$

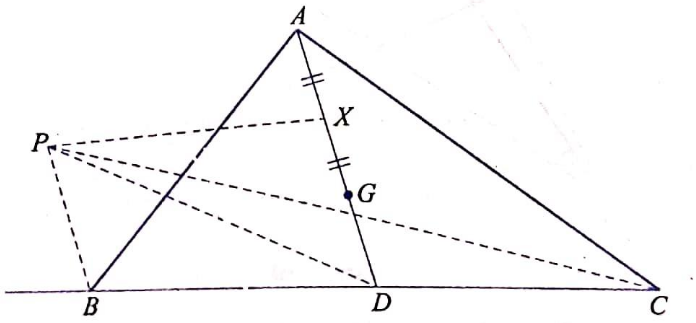


Fig. 4.38

The median PG of ΔPDX is given by

$$2PG^2 = PD^2 + PX^2 - XD^2/2 \qquad (2)$$

The median PX of ΔPAG is given by

$$2PX^2 = PA^2 + PG^2 - AG^2/2 \qquad (3)$$

Therefore (1) + ((2) × (2)) + (3) gives

$$2PD^2 + 4PG^2 + 2PX^2 = PB^2 + PC^2 - BC^2/2 + 2PD^2 + 2PX^2 - XD^2 + PA^2 + PG^2 - AG^2/2$$

Therefore, $PA^2 + PB^2 + PC^2 - 3PG^2 = BC^2/2 + XD^2 + AG^2/2$

$$= BC^2/2 + AG^2 + AG^2/2 \text{ (since } XD = AG)$$

$$= BC^2/2 + (3/2)\, AG^2$$

Similarly, if we consider the other medians BE and CF,

we get $\quad PA^2 + PB^2 + PC^2 - 3PG^2 = CA^2/2 + (3/2)BG^2$ and

$$PA^2 + PB^2 + PC^2 - 3PG^2 = AB^2/2 + (3/2)CG^2.$$

Adding we get,

$$3(PA^2 + PB^2 + PC^2 - 3PG^2)$$

$$= \frac{1}{2}(BC^2 + CA^2 + AB^2) + 3/2(BG^2 + CG^2 + AG^2)$$

$$= \frac{1}{2}(3(GA^2 + GB^2 + GC^2)) + 3/2\,(GA^2 + GB^2 + GC^2) \text{ (from Corollary 1)}$$

$$= 3(GA^2 + GB^2 + GC^2).$$

Hence $\quad PA^2 + PB^2 + PC^2 - 3PG^2 = GA^2 + GB^2 + GC^2.$ ❑

Corollary 3. If R is the circumradius and S is the circumcentre of ΔABC then

$$SG^2 = R^2 - (1/9)\,(a^2 + b^2 + c^2).$$

Proof. Take $P = S$ in Cor. 2 to get

$$SA^2 + SB^2 + SC^2 = 3R^2 = GA^2 + GB^2 + GC^2 + 3SG^2$$

$$= (1/3)(a^2 + b^2 + c^2) + 3SG^2$$

Therefore $\quad SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2).$ ❑

Theorem 29. In any ΔABC, if $\angle B > \angle C$ then the internal bisector BE of $\angle B$ is shorter than the internal bisector CF of $\angle C$.

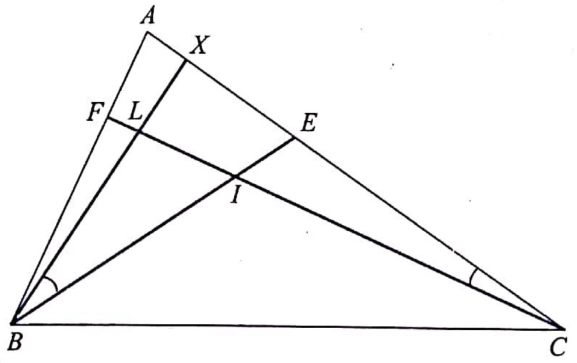


Fig. 4.39

Proof. Since $\angle B > \angle C$, $\angle ABE = \angle B/2 > \angle C/2 = \angle ACF$. Let X be the point on the segment AE such that $\angle XBE = \angle ACF$. Now, BE and CF meet at I the incentre of ΔABC. Let BX meet CF at L. By construction, ΔXBE and ΔXCL are equiangular and hence similar. Therefore $\dfrac{BE}{CL} = \dfrac{BX}{CX}$.

In ΔXBC. we have $\angle XBC = \frac{\angle B}{2} + \frac{\angle C}{2} > \frac{\angle C}{2} + \frac{\angle C}{2} = \angle XCB$ and hence $XC > BX$.

Therefore $1 < \frac{BX}{XC} = \frac{BF}{CL}$ or $BE < CL$.

Hence $BE < CL < CF$. ❑

Another proof of Theorem 29 We use again the same figure Fig. 4.39. By construction $\angle LBE = \angle LCE$ and hence the four points L, B, C, E are concylic. We have,

$$\angle C = \angle BCE < \frac{1}{2}(\angle B + \angle C) = \angle CBL < \frac{1}{2}(\angle A + \angle B + \angle C) = 90°, \text{ (Fig. 4.39)}$$

Therefore $\angle BCE < \angle CBL < 90°$ and the chords BE and CL of the circle BCE subtend different acute angles on the circumference of the circle. This implies that $BE \neq CL$; also the shorter chord being farther from the centre, subtends a smaller acute angle at the circumference. Hence $BE < CL$. But $CL < CF$ and therefore $BE < CF$. ❑

Corollary. If two internal bisectors of a triangle are equal, then the triangle is isosceles.

Proof. Immediate from the theorem. ❑

Theorem 30. The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle.

Proof. Let the external bisectors of $\angle B$ and $\angle C$ of ΔABC meet at I_a (Fig. 4.40). Then the distances of I_a from BC and AB are equal as I_a lies on the external bisector of $\angle B$. Also I_a lies on the external bisector of $\angle C$ implies that the distance of I_a from BC and CA are equal. Thus I_a is at the same distance, say r_a, from the three sides of the ΔABC. Hence I_a must lie on the internal bisector AI of $\angle A$. ❑

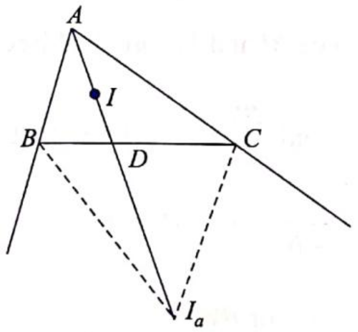


Fig. 4.40

The point I_a is called the *excentre* opposite to A. Similarly, the external bisectors of $\angle C$ and $\angle A$ meet the internal bisector of $\angle B$ at a point I_b called the excentre opposite to B; and the external bisectors of $\angle A$ and $\angle B$ meet the internal bisector of $\angle C$ at I_c, the excentre opposite to C.

We note that the incentre I is equidistant from the three sides BC, CA, AB of ΔABC. If r is the distance of I from the sides of ΔABC then the circle with centre I and radius r touches all the three sides of the triangle and is inscribed in the triangle. It is called the *incircle* of ΔABC. The circle with centre I_a and radius r_a touches the sides BC, CA, AB of ΔABC. It touches BC at a point on the line segment BC, whereas it touches the other two sides CA, AB at points on CA, AB produced. The circles (I_a, r_a), (I_b, r_b), (I_c, r_c) are the three escribed circles known as the *excircles* opposite to A, B and C respectively. (See Fig. 4.41).

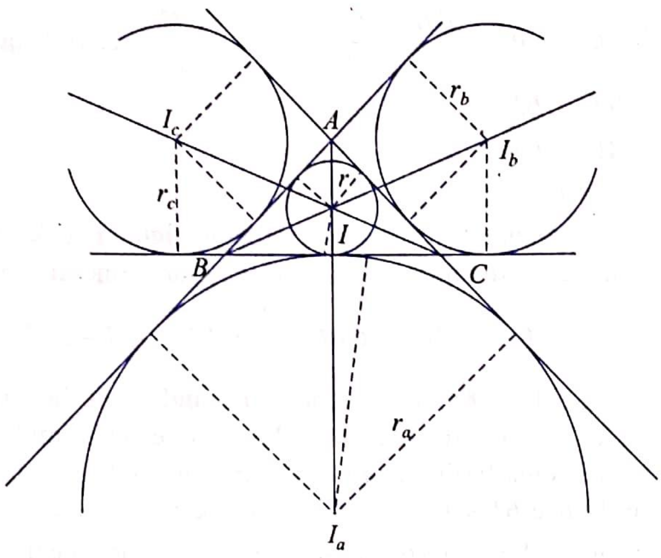


Fig. 4.41

Theorem 31. The incentre I and the excentre I_a opposite to A divide the bisector AU harmonically, where U is the point of intersection of the internal bisector of $\angle A$ and BC.

Proof. It is required to prove that $\frac{AI}{IU} = \frac{AI_a}{UI_a}$. Fig. 4.42.

Consider ΔBAU. By Theorem 31 of Chapter 3 we have

$$\frac{AI}{IU} = \frac{BA}{BU} = \frac{AI}{UI_a} \text{ since } BI \text{ and } BI_a \text{ are the bisectors of } \angle ABU \text{ of } \Delta ABC.$$

In fact, we have $\frac{BU}{UC} = \frac{c}{b}$ and $\frac{BV}{CV} = \frac{c}{b}$ (Fig. 4.42). ❑

$$\frac{BU}{BU + UC} = \frac{c}{c+b} \quad \text{or} \quad BU = BC \cdot \frac{c}{c+b} = \frac{ac}{b+c}.$$

$$\frac{BV}{CV - BV} = \frac{c}{b-c} \text{ or } BV = BC \cdot \frac{c}{b-c} = \frac{ac}{b-c}.$$

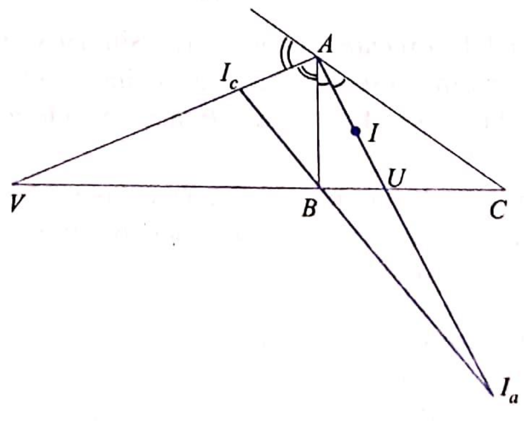


Fig. 4.42

$$\frac{AI}{IU}=\frac{BA}{BU}=\frac{c}{ac/(b+c)}=\frac{b+c}{a} \quad \text{and}$$

$$\frac{AI_c}{I_cV}=\frac{BA}{BV}=\frac{c}{ac/(b-c)}=\frac{b-c}{a}.$$

Theorem 32. If I is the incentre of ΔABC and I_a is the excentre opposite to A then $AI \cdot AI_a = AB \cdot AC$.

Proof. As $\angle IBI_a = \angle ICI_a = 90°$, the circle on II_a as diameter passes through B and C. If AB cuts this circle again at B_1 (Fig. 4.43) then $AB_1 = AC$ (Why?).

Therefore $\quad AI \cdot AI_a = AB \cdot AB_1 = AB \cdot AC.$ ❑

Theorem 33. If the incircle of ΔABC touches the sides BC, CA, AB of the triangle at X, Y, Z then $BX = s - b$, $CY = s - c$ and $AZ = s - a$ where $2s = a + b + c$, the perimeter of ΔABC.

Proof. We have $\quad AZ = AY$, $BZ = BX$ and $CX = CY$. (Fig. 4.44).

Therefore $\quad BZ + BX = BA + BC - AZ - CX$

$= BA + BC - AY - CY$

$= c + a - b.$

Therefore $\quad 2BZ = 2s - 2b$ or $BZ = s - b = BX$

Similarly we get $\quad CY = s - c$ and $AZ = s - a.$ ❑

Theorem 34. If the escribed circle opposite to A touches the sides BC, CA, AB of ΔABC at X_a, Y_a, Z_a respectively, then $AZ_a = AY_a = s = BX_b = BZ_b = CY_c = CX_c$ (with obvious meanings for X_b, Y_b, Z_c, X_c, Y_c, and Z_c).

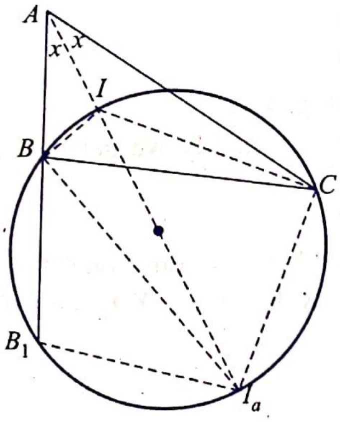


Fig. 4.43

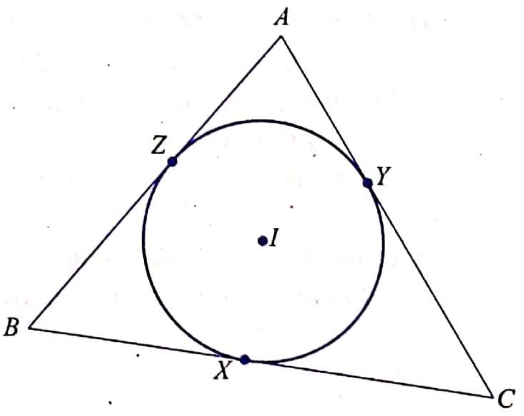


Fig. 4.44

Proof. We have $AY_a = AZ_a$, $BX_a = BZ_a$ and $CY_a = CX_a$, being the tangents from A, B, C to the escribed circle (I_a, r_a).

Therefore $\quad 2AZ_a = AZ_a + AY_c$

$= AB + AC + BZ_a + CY_a = AB + AC + BX_a + CX_a$

$= AB + AC + BC = 2s.$ ❑

Therefore $\quad AZ_a = AY_a = s$. Similarly we get $BX_b = BZ_b = s = CY_c = CX_c$. ❑

Corollary. $\quad BX_a = BZ_a = s - c$; $CX_a = CY_a = s - b$

Proof. We have $\quad BX_a = BZ_a = AZ_a - AB = s - c$ and

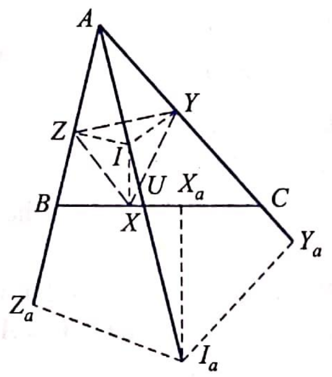


Fig. 4.45

$$CX_a = CY_a = AY_a - AC = s - b.$$ ❑

Definition 3. Two points on a side of a triangle are *isotomic points* if they are equidistant from the midpoint of this side.

Theorem 35. The points of contact of a side of a triangle with the incircle and excircle corresponding to this side are two isotomic points.

Proof. In our usual notations, we wish to prove that

$$BX = CX_a,\ AY = CY_b \text{ and } AZ = BZ_c. \text{ (Fig. 4.41).}$$

We have $BX = s - b$ and $CX_a = s - b$ (Theorem 33 and Corollary to Theorem 34). This means that X and X_a are isotomic points. Similarly, Y and Y_b are isotomic points on CA and so are Z and Z_c on AB. ❑

Corollary.

(*i*) $XX_a = b - c,\ YY_b = a - c,\ ZZ_c = a - b$

(*ii*) $ZZ_a = YY_a = a;\ ZZ_b = XX_b = b;\ YY_c = XX_c = c.$

Proof. (*i*) $XX_a = BC - BX - CX_a = a - 2(s - b) = b - c.$

Similarly, $YY_b = a - c$ and $ZZ_c = a - b$. (Fig. 4.41).

(*ii*) $ZZ_a - ZB + BZ_a = (s - b) + (s - c) = a$. Similarly we get the other equations. ❑

Theorem 36.

(1) X_c and X are isotomic points on BC; Y_a and Y_c are isotomic on CA and Z_a and Z_b are isotomic points on AB. Further $X_bX_c = b + c$, $Y_cY_a = c + a$ and $Z_aZ_b = a + b$.

(2) $Y_bY_c = Z_bZ_c = a$.

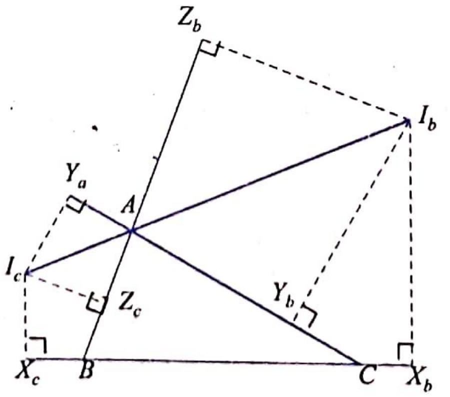


Fig. 4.46

Proof. 1. We have $BX_c = CX_c - a = s - a$ and $CX_b = BX_b - a = s - a$. Therefore X_b and X_c are isotomic points on BC. Furthermore, $X_bX_c = X_bC + CB + BX_c = (s - a) + a + (s - a) = 2s - a = b + c$. Similarly we get the corresponding relations for the other sides.

2. $Y_bY_c = CY_c - CY_b = s - CY_b = s - CX_b = s - (s - a) = a$. Similarly, $Z_bZ_c = a$. ❑

Theorem 37.

(1) For any triangle the area is equal to the product of the inradius and the semiperimeter. *i.e.*, for any ΔABC,

area of $\Delta ABC = \Delta = rs$

(2) $\Delta = r_a(s - a) = r_b(s - b) = r_c(s - c)$.

Proof. (1) If I is the incentre of ΔABC, we have

Area of ΔABC = area of ΔBIC + area ΔCIA + area ΔAIB. (Fig. 4.41)

$$= (1/2)\ ar + (1/2)\ br + (1/2)\ cr = rs$$

Thus $\Delta = rs$.

(2) We see from Fig. 4.41 that

Area ΔABC = area ΔABI_a + area ΔACI_a – area ΔBCI_a

$$= (1/2)cr_a + (1/2)br_a - (1/2)ar_a = (1/2)(b + c - a)r_a$$

$$= (s - a)r_a$$

Thus $\Delta = (s - a)r_a$. Similarly we obtain

$$\Delta = (s - b)r_b,$$

$$\Delta = (s - c)r_c.$$ ❑

Theorem 38. $rr_a = (s - b)(s - c)$

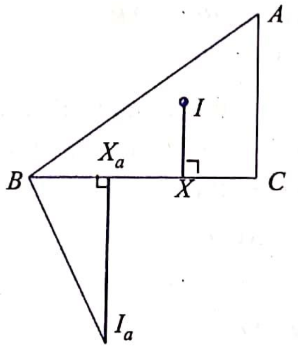


Fig. 4.47

Proof. We have $\angle IBX = \angle IBI_a - \angle I_aBX_a$

$$= 90° - \angle I_aBX_a = \angle BI_aX_a$$

Therefore the right triangles IBX and BI_aX_a are similar,

and
$$\frac{IX}{BX} = \frac{BX_a}{I_aX_s} \text{ or } \frac{r}{s-b} = \frac{s-c}{r_a}$$

i.e.,
$$rr_a = (s - b)(s - c)$$ ❑

Theorem 39. Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (known as Hero's formula)

Proof. We have seen that $r = \Delta/s$ and $r_a = \Delta/s - a$

Therefore $rr_a = \dfrac{\Delta^2}{s(s-a)}$. But by Theorem 38 $rr_a = (s-b)(s-c)$.

This gives $\dfrac{\Delta^2}{s(s-a)} = (s-b)(s-c)$ or $\Delta^2 = s(s-a)(s-b)(s-c)$. ❑

Corollary. (1) $rr_a r_b r_c = \Delta^2$

(2) $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$.

Proof.

$$(1)\ rr_a r_b r_c = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}$$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \Delta^2 \text{ from the theorem.}$$

$$(2)\ \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{(s-c)}{\Delta} + \frac{(s-b)}{\Delta} + \frac{(s-c)}{\Delta}$$

$$= \frac{3s - (a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$ ❑

Theorem 40. For any triangle ABC, if the incircle touches the sides BC, CA, AB at X, Y, Z respectively, then

$$\frac{\text{area of } \Delta ABC}{\text{area of } \Delta XYZ} = \frac{2R}{r}$$

where R is the circumradius and r is the inradius of ΔABC.

Proof. The quadrilateral $AZIY$ is cyclic (Why?) See Fig. 4.48.

Therefore $\angle BAC + \angle YIZ = 180°$

So, by Theorem 36 of Chpater 3, we get

$$\frac{\Delta IYZ}{\Delta ABC} = \frac{IY \cdot IZ}{AB \cdot AC} = \frac{r^2}{bc} = \frac{cr^2}{abc}.$$

Similarly $\dfrac{\Delta IZX}{\Delta ABC} = \dfrac{br^2}{abc}$ and $\dfrac{\Delta IXY}{\Delta ABC} = \dfrac{ar^2}{abc}$.

Adding we get $\dfrac{\Delta XYZ}{\Delta ABC} = \dfrac{r^2(a+b+c)}{abc} = \dfrac{2sr^2}{abc} = \dfrac{2\Delta r}{4R\Delta} = \dfrac{r}{2R}$.

(From Theorem 37 and Cor. to Thm. 21 of Ch. 4)

Thus $\dfrac{\Delta ABC}{\Delta XYZ} = \dfrac{2R}{r}$. ❑

Theorem 41. $\dfrac{1}{r} = \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}$

where r is the inradius and h_a, h_b, h_c are the altitudes through A, B, C of ΔABC.

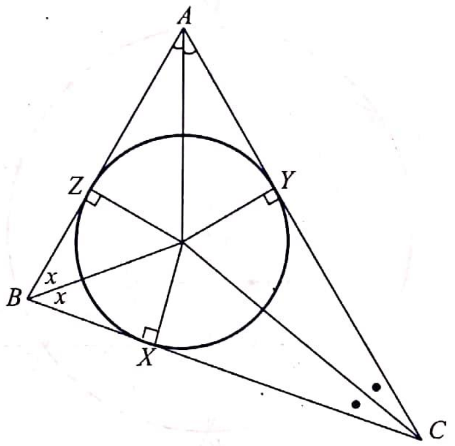


Fig. 4.48

Proof. We have $\Delta = (1/2)ah_a = (1/2)bh_b = (1/2)ch_c$.

Therefore $2\Delta = ah_a = bh_b = ch_c$; which gives

$$\frac{2s}{(1/r)} = \frac{a}{(1/h_a)} = \frac{b}{(1/h_b)} = \frac{c}{(1/h_c)} = \frac{a+b+c}{(1/h_a + 1/h_b + 1/h_c)}$$

Hence $\dfrac{1}{r} = \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}$. ❑

Corollary. $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}$.

Proof. Follows immediately from the Corollary to Theorem 39 and Theorem 41. ❑

Theorem 42. $r_a + r_b + r_c = r + 4R$.

Proof. Let the circumcircle of ΔABC meet the line AII_a at P. Then the centre of the circle IBI_aC on II_a as diameter lies on II_a as well as the perpendicular bisector of BC. The arc BP and the arc PC make equal angles at A implies that P is the midpoint of arc BC. Hence P lies on the perpendicular bisector of BC. This means that P must be the midpoint of II_a. Let PQ be the circumdiameter of ΔABC through P. Let IK be parallel to BC meeting I_aX_a at K (Fig. 4.49) and meeting SD at L. Then we have,

$$PD + DL = PL = (1/2)I_aK \text{ (since } PL \parallel I_aK) = (1/2)(I_aX_a + X_aK)$$

Also $DL = X_aK = r$. (Fig. 49).

Therefore $PD = (1/2)(I_aX_a + r) - r = (1/2)(r_a - r)$ (since $I_aX_a = r_a$).

Just as we observed that P is the midpoint of II_a, we can prove that Q is the midpoint of I_bI_c. Also,

$$DQ = (1/2)(I_bX_b + I_cX_c) = (1/2)(r_b + r_c).$$

$\therefore$ $2R = PQ = PD + DQ = (1/2)(r_a + r_b + r_c - r)$

and hence $r_a + r_b + r_c = 4R + r$. ❑

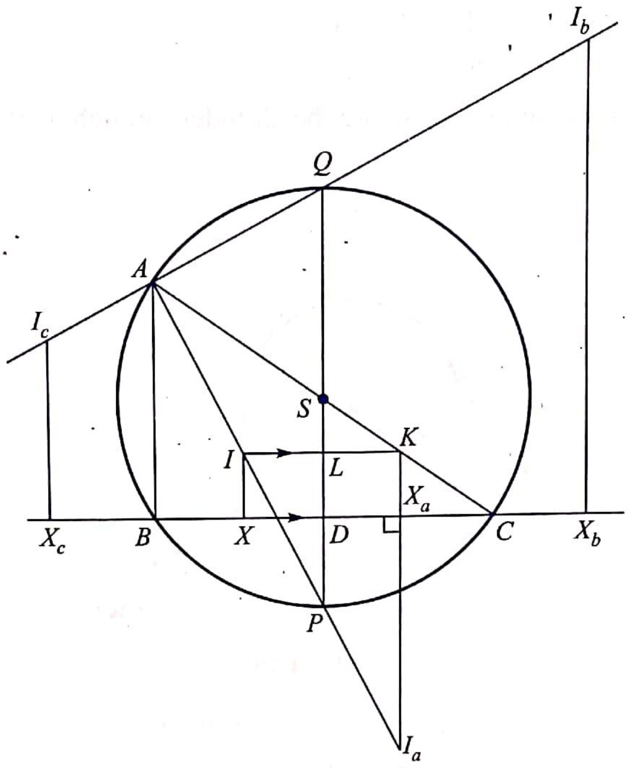


Fig. 4.49

Corollary. $SD + SE + SF = R + r$ (Fig. 4.49).

Proof. $SD = SP - DP = R - DP = R - (1/2)(r_a - r)$

(as in the proof of Theorem 42)

Similarly $SE = R - (1/2)(r_b - r)$ and $SF = R - (1/2)(r_c - r)$

Therefore $SD + SE + SF = 3R - (1/2)(r_a + r_b + r_c - 3r)$.

$= 3R - (1/2)(r + 4R - 3r) = R + r$. ❑

Theorem 43. (Euler's Theorem).

If d is the distance between the circumcentre and the incentre of a triangle then $SI^2 = d^2 = R^2 - 2Rr$.

Proof. We have $AI \cdot IP = IJ \cdot IK = (R + d)(R - d) = R^2 - d^2$. (Fig. 4.50).

As we have seen in the proof of Theorem 42, P is the centre of the circle IBI_aC and hence $PI = PB = PC$. Therefore $AI \cdot PC = R^2 - d^2$. Now, the right triangles AIZ and QPC are similar, since $\angle IAZ = \angle PAB = \angle PQC$ (Why?). This gives $AI/PQ = IZ/PC$ or $AI \cdot PC = PQ \cdot IZ$. Thus $R^2 - d^2 = 2Rr$ or $d^2 = R^2 - 2Rr$. ❑

Theorem 44. $(SI_a)^2 = R^2 + 2Rr_a$, $(SI_b)^2 = R^2 + 2Rr_b$ and $(SI_c)^2 = R^2 + 2Rr_c$

Proof. Exercise. ❑

Corollary. $(II_a)^2 = 4R(r_a - r)$, $(I_bI_c)^2 = 4R(r_b + r_c)$.

Proof. We use Fig. 4.49. For the triangle SII_a, SP is a median.

Therefore, $SI^2 + SI_a^2 = 2SP^2 + (1/2)II_a^2$.

This gives $R^2 - 2Rr + R^2 + 2Rr_a = 2R^2 + (1/2)\, II_a^2$

Hence $II_a^2 = 4R(r_a - r)$

Similarly, for the $\Delta SI_b I_c$, SQ is a median and so

$$SI_b^2 + SI_c^2 = 2SQ^2 + (1/2)I_bI_c^2$$

Therefore $R^2 + 2Rr_b + R^2 + 2Rr_c = 2R^2 + (1/2)\, I_bI_c^2$

Hence $$I_bI_c^2 = 4R(r_b + r_c)$$ ❑

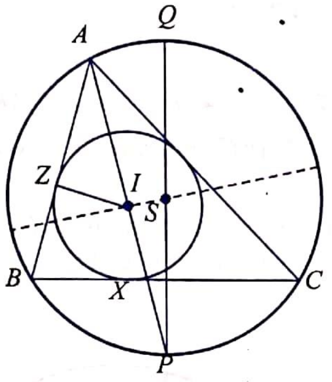


Fig. 4.50

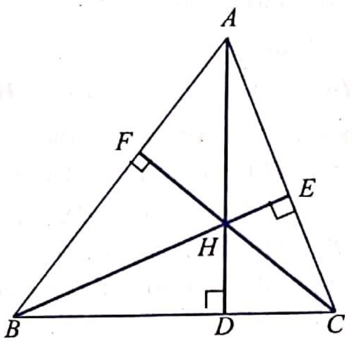


Fig. 4.51

Theorem 45. If the line of centres for the two circles (S, R) and (I, r) satisfy $SI^2 = R^2 - 2Rr$ then an infinite number of triangles may be found such that (I, r) is the incircle of each one of them and (S, R) is the circumcircle of each one of them.

Proof. From any point A on the circles (S, R) draw the two tangents AB, AC to the other circle (I, R) meeting (S, R) again at B, C (Fig. 4.50). If AI meets the circle (S, R) at P, then $SI^2 = R^2 - 2Rr$ implies that $AI \cdot IP = 2Rr$ (Why?). If Z is the foot of the perpendicular from I on AB, then the triangles AIZ and QPC are similar. Therefore $AI \cdot PC = PQ \cdot IZ = 2Rr$. This means that $AI \cdot IP = AI \cdot PC = 2Rr$ and hence $IP = PC$. As we have already seen in the proof of Theorem 42, for ΔABC, the point P must be the midpoint of the line joining the incentre and the excentre opposite to A; and we have $PB = PC = PI_a$. Now I lies on the internal bisector of $\angle A$ (Fig. 4.50) and $PC = PI$ implies that I must be the incentre of ΔABC. Since we have infinitely many choices available for A, there is an infinity of triangles having (I, r) as the incircle and (S, R) as the circumcircle. ❑

Theorem 46. Let ABC be a triangle with AD, BE, CF as the altitudes and H the orthocentre. Then $AH \cdot HD = BH \cdot HE = CH \cdot HF$.

Proof. The right triangles BHF and CHE are similar since $\angle FBH = \angle ECH$ (the quadrilateral $BCEF$ is cyclic) (Fig. 4.51). Therefore,

$$\frac{HF}{BH} = \frac{HE}{CH} \quad \text{or} \quad CH \cdot HF = BH \cdot HE$$

Similarly, $AH \cdot HD = BH \cdot HE$ ❑

Note. The same proof works when ΔABC is obtuse angled and H lies outside ΔABC.

Corollary. $HA \cdot HD = (1/2)(a^2 + b^2 + c^2) - 4R^2$.

Proof. We have $HA \cdot HD = HB \cdot HE =$ Power of H with respect to the circle on BC as diameter. (Fig. 4.52). Therefore, we get

$$HA \cdot HD = (a/2)^2 - A'H^2 \qquad (1)$$

Now, HA' is the median through H for ΔHBC. Therefore,

$$HB^2 + HC^2 = (1/2)a^2 + 2(HA')^2 \qquad (2)$$

If AP is the diameter of the circumcircle through A, then $CP \perp AC$ and hence $BH \parallel PC$. Similarly, $CH \parallel PB$. This means that $BPHC$ is a parallelogram and $BH = PC$. This gives,

$$\left\{\begin{array}{ll} & HB^2 = PC^2 = AP^2 - AC^2 = 4R^2 - b^2 \\ \text{Similarly} & HC^2 = PB^2 = AP^2 - AB^2 = 4R^2 - c^2 \end{array}\right\} \qquad (3)$$

From (1), (2) and (3) we get,

$$HA \cdot HD = a^2/4 - (HB^2/2 + HC^2/2 - a^2/4) = a^2/2 + b^2/2 + c^2/2 - 4R^2 \quad \square$$

Theorem 47. The chord CX of the circumcircle of ΔABC perpendicular to BC is equal to AH, where H is the orthocentre of ΔABC.

Proof. $CX \perp BC$ implies that BX is a diameter of the circumcircle of ΔABC. Therefore $\angle BAX = 90°$. This means that $AX \perp AB$ and so $AX \parallel HC$. Similarly, $AH \parallel XC$. Therefore $AHCX$ is a parallelogram and $CX = AH$. ❑

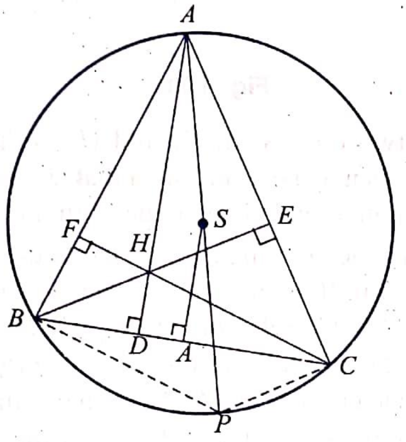


Fig. 4.52

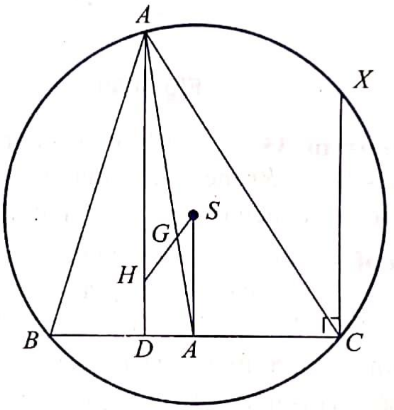


Fig. 4.53

Corollary. $AH = 2SA'$.

Proof. In ΔBCX, S and A' are the midpoints of the sides BX and BC respectively. Therefore $2SA' = CX$ and hence by Theorem 47, $AH = CX = 2SA'$. ❑

Theorem 48. In any triangle, the circumcentre, the orthocentre and the centroid are collinear. The centroid G trisects the line joining the circumcentre and the orthocentre. (This line is called the *Euler line* of the triangle).

Proof. Let the median AA' of ΔABC meet the line SH at G, where S is the circumcentre and H is the orthocentre of ΔABC (Fig. 4.53). The triangle AHG is similar to the triangle $A'SG$ and therefore,

$$\frac{AH}{SA'} = \frac{HG}{SG} = \frac{AG}{GA'}$$

But $2SA' = AH$ (Corollary to Theorem 47) and hence $AG/GA' = 2$. This means that G must be the centroid of ΔABC. Again, the above equation tells us that G trisects SH. Hence the theorem. ❑

Corollary 1.

(1) $SH^2 = 9R^2 - (a^2 + b^2 + c^2)$

(2) $GH^2 = 4R^2 - (4/9)(a^2 + b^2 + c^2)$

Proof. We have $SH = 3SG$ and $GH = 2SG$. By Cor. 3 to Theorem 28, Ch. 4 we have, $SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$.

Therefore, $SH^2 = 9SG^2 = 9R^2 - (a^2 + b^2 + c^2)$ and

$GH^2 = 4SG^2 = 4R^2 - (4/9)(a^2 + b^2 + c^2)$. ❑

Corollary. $2\ HA^2 + HB^2 + HC^2 = 12R^2 - (a^2 - b^2 + c^2)$.

Proof. By Cor. 2, Thm. 28, Chapter 4, we get

$$HA^2 + HB^2 + HC^2 = GA^2 + GB^2 + GC^2 + 3GH^2$$

$$= \frac{a^2 + b^2 + c^2}{3} + 3GH^2 \quad \text{(Cor. 1 to Thm. 28)}$$

$$= \frac{a^2 + b^2 + c^2}{3} + 12R^2 - (4/3)\,(a^2 + b^2 + c^2)$$

$$= 12R^2 - (a^2 + b^2 + c^2)$$ ❑

Note. We have already seen in the proof of Corollary to Theorem 46 that $HB^2 = 4R^2 - b^2$, $HC^2 = 4R^2 - C^2$ and $HA^2 = 4R^2 - a^2$. Therefore, $HA^2 + HB^2 + HC^2 = 12R^2 - (a^2 + b^2 + c^2)$.

Theorem 49. If the altitude AD of ΔABC meets the circumcircle again at D', then D is the midpoint of HD' where H is the orthocentre of ΔABC. In other words, the line segment of the altitude extended between the orthocentre and the other point of intersection with the circumcircle is bisected by the corresponding side of the triangle.

Proof. We have BH and HD perpendicular to CA and BC respectively, and so $\angle BHD = \angle ACB$. Also, $\angle HD'B = \angle AD'B = \angle ACB$ (angles in the same segment). Therefore $\angle BHD = \angle HD'B$ and so $BH = BD'$. Now $BD \perp HD'$ implies that $HD = DD'$. ❑

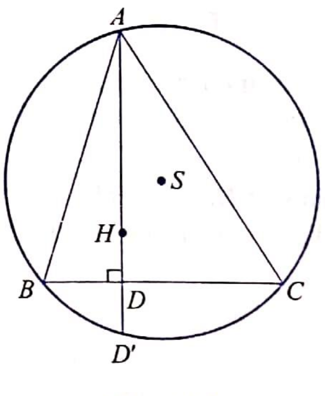


Fig. 4.54

Corollary 1. $BD \cdot DC = AD \cdot HD$

Proof. By the secant property of the circumcircle we get $BD \cdot DC = AD \cdot DD' = AD \cdot HD$ (by Theorem 49). ❑

Corollary 2. The circumcircle of ΔHBC is equal to the circumcircle of ΔABC (*i.e.*, they have the same radius).

Proof. The triangles HBC and $D'BC$ are congruent (Why?). Therefore, their circumcircles are equal circles. But the circumcircle of $\Delta D'BC$ is the same as that of ΔABC. Hence the result. ❑

Definition 4. If D, E, F are the feet of the altitudes of a triangle ABC on the corresponding sides then ΔDEF is *the orthic triangle* (or *the Pedal triangle*) of ΔABC.

Theorem 50. The three triangles cut off from a given triangle by the sides of its orthic triangle and the given triangle itself are mutually similar.

Proof. The quadrilateral $BCEF$ (Fig. 4.55) is cyclic (Why?) and hence $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Therefore $\Delta AEF \,|||\, \Delta ABC$. Likewise one can prove that $\Delta BDF \,|||\, \Delta BAC$ and $\Delta CDE \,|||\, \Delta CAB$. ❑

Theorem 51. A is the midpoint of the arc $F'E'$ of the circumcircle of ΔABC; B is the midpoint of the arc $F'D'$ and C is the midpoint of the arc $B'E'$, where D', E', F' are the points where the altitudes AD, BE, CF meet the circumcircle.

Proof. We note that $\angle ABE = \angle FBE = \angle FCE = \angle F'CA$ since the quadrilateral $BCEF$ is cyclic. Therefore arc $F'A$ = arc AE' or A is the midpoint of the arc $F'E'$ of the circle ABC. Similarly, we get the other results. ❑

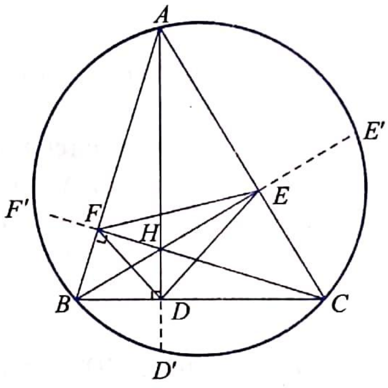


Fig. 4.55

Corollary. The radii of the circumcircle through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle. In other words $SA \perp EF$, $SB \perp FD$ and $SC \perp DE$ (Fig. 4.55).

Proof. E and F are the midpoints of HE' and HF' (Thm 49) and therefore $EF \parallel E'F'$ and $EF = (1/2)E'F'$. Now by Theorem 51, SA bisects the chord $E'F'$ and $SA \perp E'F'$. This implies that $SA \perp EF$. ❑

Theorem 52. The orthocentre of an acute angled triangle is the incentre of the orthic triangle.

Proof. We again use Fig. 4.55. The line BE' bisects $\angle F'E'D'$ since arc BF' = arc BD'. Now $EF \parallel E'F'$ and $ED \parallel E'D'$ implies that BE bisects $\angle DEF$ of the orthic triangle. Similarly AD bisects $\angle FDE$ and CF bisects $\angle EFD$. Therefore, H is the incentre of ΔDEF. ❑

Corollary. The sides of a triangle bisect externally the angles of its orthic triangle.

Proof. The sides of a triangle ABC are perpendicular to the altitudes, which are the internal bisectors of the angles of the orthic triangle. Hence the sides of ΔABC are the external bisectors of the angles of its orthic triangle ΔDEF. ❑

Note. We observe that the quadrilaterals $BDHF$, $HDCE$, and $BCEF$ are all cyclic. Therefore, we get

$$\angle FDH = \angle FBH = \angle FBE = \angle FCE = \angle HCE = \angle HDE$$

(Angles in the same segment).

$\therefore$ DH bisects $\angle FDE$ of ΔDEF. Similarly, EH bisects $\angle DEF$ and hence H is the incentre of ΔDEF.

Theorem 53. With the usual notations, $AH + r_a = BH + r_b = CH + r_c = 2R + r$.

Proof. As in the proof of Theorem 42, we have $r_a - r = 2(SP - SA')$ where P is the point where SA' and AI meet on the circumcircle. But $2SA' = AH$ (Cor. to Theorem 47).

$\therefore$ $AH + r_a - 2SP + r = 2R + r$. Similarly we get

$$BH + r_b = 2R + r = CH + r_c.$$ ❑

Theorem 54. If ABC is an acute-angled triangle and DEF is its orthic triangle then

$$\frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} = \frac{R+r}{r}.$$

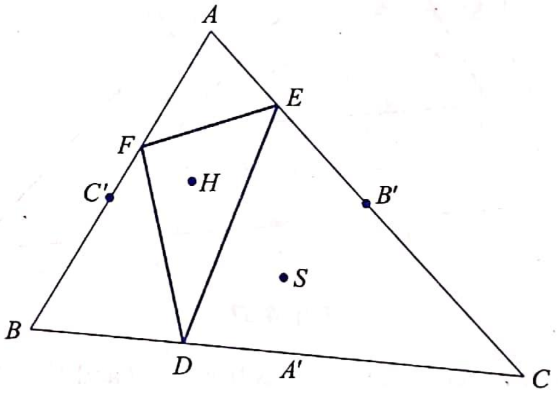


Fig. 4.56

Proof. The triangles AEF and ABC are similar.

$$\therefore \quad \frac{EF}{BC} = \frac{\text{Circumference of } \Delta AEF}{\text{Circumference of } \Delta ABC} = \frac{AH}{2R} = \frac{2SA'}{2R} = \frac{SA'}{R}$$

Similarly $$\frac{FD}{CA} = \frac{SB'}{R}; \quad \frac{DE}{AB} = \frac{SC'}{R}$$

Adding $$\frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} = \frac{SA' + SB' + SC'}{R} = \frac{R+r}{R}.$$ ❑

Theorem 55. Perimeter of the orthic triangle of ΔABC is

$$\frac{2 \text{ area of } \Delta ABC}{R}.$$

Proof. As in the proof of Theorem 54,

$$EF = \frac{SA'}{R} BC, \; FD = \frac{SB'}{R} CA \text{ and } DE = \frac{SC'}{R} AB$$

$$\therefore \quad \text{Perimeter of } \Delta DEF = \frac{1}{R}(SA' \,.\, BC + SB' \,.\, CA + SC' \,.\, AB)$$

$$= \frac{1}{R} 2(\Delta BSC + \Delta CSA + \Delta ASB)$$

$$= \frac{2}{R} \text{ area of } \Delta ABC.$$ ❑

Theorem 56. (The Nine-Point Circle Theorem).

The feet of the three altitudes of any triangle, the midpoints of the three sides and the midpoints of the segments from the orthocentre to the three vertices all lie on a circle of radius equal to half the circumradius. Further the centre of this circle bisects the line joining the orthocentre and the circumcentre.

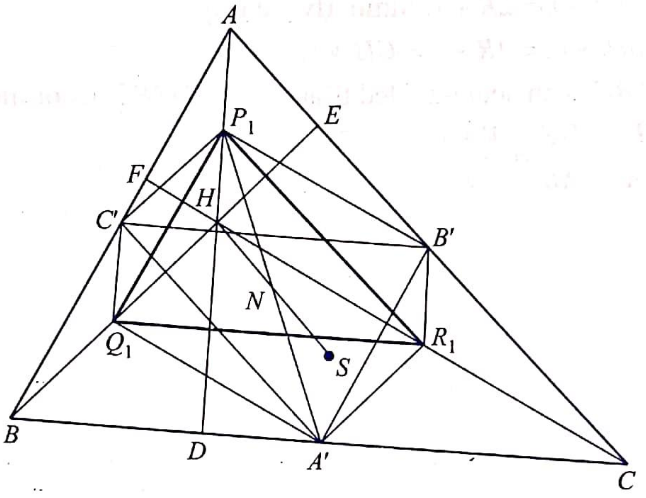


Fig. 4.57

Proof. Let D, E, F be the feet of the altitudes from A, B and C of ΔABC; let A', B', C' be the midpoints of BC, CA, AB respectively; H be the orthocentre, S the circumcentre and P_1, Q_1, R_1 be the midpoints of AH, BH and CH respectively. It is required to prove that D, E, F, A', B', C', P_1, Q_1, R_1 all lie on a circle with centre N the midpoint of the segment SH, and radius $R/2$. We have $C'B' \parallel BC$ and $Q_1R_1 \parallel BC$, since Q_1 and R_1 are the midpoints of the sides HB, HC of ΔHBC, Also $C'B' = Q_1R_1 = (1/2)BC$, Again from ΔABH, we get $C'Q_1 \parallel AH$ and $C'Q_1 = (1/2)\,AH$.

From ΔACH, we get $B'R_1 \parallel AH$ and $B'R_1 = (1/2)AH$.

Thus $C'Q_1 \parallel B'R_1$ and $C'Q_1 = B'R_1 = (1/2)AH$.

AH is perpendicular to BC implies that $C'Q_1$ is perpendicular to Q_1R_1. $\therefore$ $B'C'Q_1R_1$ is a rectangle. Similarly $C'A'R_1P_1$, and $A'B'P_1Q_1$ are also rectangles. Hence P_1, C', Q_1. A', R_1, B' all lie on the circle having $A'P_1$, $B'Q_1$ and $C'R_1$ as diameters.

Also $\angle A'DP_1 = 90°$ implies that this circle also passes through D. Similarly, it passes through E and F as well. Thus the nine-points A', B', C', P_1, Q_1, R_1, D, E, F, all lie on a circle. The radius of this nine-point circle is $(1/2)\,A'P_1$.

From the parallelogram $AP_1A'S$ we get $R = SA = A'P_1$

$\therefore$ The radius of the nine-point circle is $R/2$.

It remains to prove that the centre N of the nine-point circle is the midpoint of SH. We have (A', P_1), (B', Q_1), (C', R_1) as three pairs of diametrically opposite points on the nine-point circle. Therefore the triangle $A'B'C'$ can be obtained from $\Delta P_1Q_1R_1$ by a half turn (rotation through 180°) about the centre of the nine-point circle. Under this rotation, the triangles $A'B'C'$ and $P_1Q_1R_1$ get interchanged and hence their orthocentres also get interchanged. The orthocentre of $\Delta P_1Q_1R_1$ is H and the orthocentre of $\Delta A'B'C'$ is S.

Therefore H and S correspond with respect to a half turn about the centre of the nine-point circle. Hence the midpoint N of HS is the centre of the nine-point circle. ❑

Note. 1. The circumcentre, orthocentre, centroid and the nine-point centre all lie on the Euler line. G trisects SH and N bisects SH. S and N divide GH harmonically in the ratio 2 : 1.

2. The fact that the nine-point centre bisects SH can also be seen as follows. $P_1HA'S$ is a parallelogram and hence the diagonals bisect each other. So the midpoint of SH must be the midpoint of $A'P_1$, which is the centre of nine-point circle of ΔABC.

Theorem 57. The sum of the powers of the vertices of a triangle ABC, with respect to its nine-point circle is $(1/4)\ (a^2 + b^2 + c^2)$.

Proof. The power of A with respect to the nine-point circle is

$$AE \cdot AB' = AF \cdot AC' \text{ (Fig. 4.57).}$$

$\therefore$ Power of A with respect to the nine-point circle is

$$= (1/2)\ (AE \cdot AB' + AF \cdot AC') = (1/2)\left(AE \cdot \frac{b}{2} + AF \cdot \frac{c}{2}\right)$$

$$= (1/4)\ (b\,AE + c\,AF)$$

The sum of the powers of A, B, C with respect to the nine-point circle

$$= (1/4)\ \{(b\,AE + c\,AF) + (c\,BF + a\,BD) + (a\,DC + b\,CE)\}$$

$$= (1/4)\ (b^2 + c^2 + a^2).$$ ❑

Corollary. $NA^2 + NB^2 + NC^2 + NH^2 = 3R^2$

Proof. Power of A with respect to the nine-point circle is

$$AN^2 - \left(\frac{R}{2}\right)^2 = AN^2 - \frac{R^2}{4}$$

Sum of the powers of the vertices with respect to the circle $(N, R/2)$

$$= (1/4)\ (a^2 + b^2 + c^2)$$

$$= NA^2 + NB^2 + NC^2 - 3\frac{R^2}{4}.$$

$$\therefore \quad NA^2 + NB^2 + NC^2 + NH^2 = (1/4)(a^2 + b^2 + c^2) + 3\ \frac{R^2}{4} + \frac{SH^2}{4}$$

$$= \frac{1}{4}\ (a^2 + b^2 + c^2) + \frac{3R^2}{4} + \frac{9R^2 - (a^2 + b^2 + c^2)}{4} = 3R^2$$

(From Cor. 1 to Theorem 48) ❑

Theorem 58. All triangles inscribed in a given circle and having a given point as the orthocentre, have the same nine-point circle.

Proof. All these triangles have the same circumcentre S and the same orthocentre H. Therefore they have the same nine-point centre N, namely the midpoint of SH. Further the radius of the nine-point circle is $R/2$ where R is the radius of the given circle. ❑

Theorem 59 (Feuerbach's Theorem). In any triangle, the nine-point circle touches the incircle and the three described circles.

Proof.

First Proof. We assume $\angle B > \angle C$. Let $A'T$ be the tangent (Fig. 4.58) to the nine-point circle at A'. Then we have $\angle CA'T = \angle CA'B' - \angle B'A'T = \angle B - \angle B'C'A'$ (angle between

a chord and a tangent = angle in the alternate segment). $= \angle B - \angle C$ (Note that $\Delta A'B'C'$ is the medial triangle) (1)

Let AI meet BC at U. We have $\frac{IU}{UI_a} = \frac{r}{r_a}$ (Fig. 4.59)

$\therefore$ The transverse common tangents to the incircle and the excircle (I_a, r_a) meet at U (Theorem 14).

Hence if UP_1 and UP_1' are the tangents to the incircle and the excircle opposite to A from U then P_1UP_1' is a straight line.

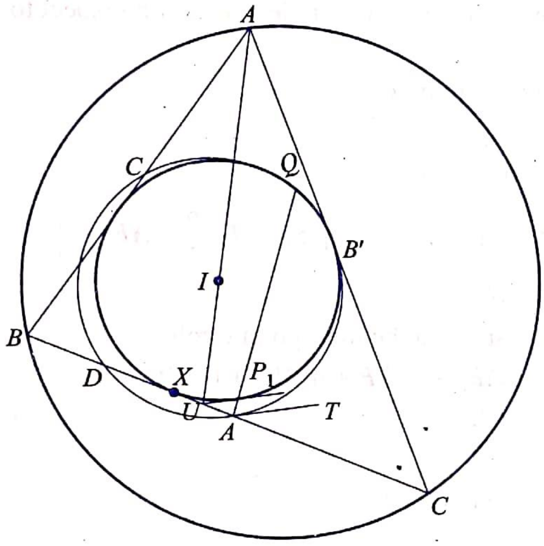


Fig. 4.58

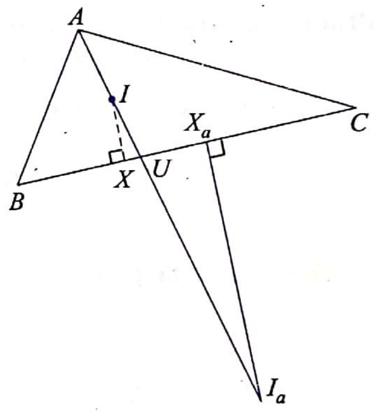


Fig. 4.59

$$\angle AUP_1 = \angle BUA \text{ (since } UP_1, UB \text{ are tangents to the in circle).}$$

$$= \angle CAU + \angle BCA = \frac{\angle A}{2} + \angle C.$$

$$\angle P_1UC = 180° - \angle BUP_1 = 180° - 2\angle BUA$$

$$= 180° - (\angle A + 2\angle C) = \angle B - \angle C \quad (2)$$

$\therefore$ $UP_1 \parallel A'T$ from (1) and (2).

Let $A'P_1$ meet the incircle again at Q. AU divides I_a harmonically implies that the feet of the perpendiculars on BC from A, U divide the feet of the perpendiculars on BC from I, I_a harmonically.

$\therefore$ DU divides XX_a harmonically and A' is the midpoint of XX_a implies that

$$A'P_1 \cdot A'Q = A'X^2 = A'U \cdot A'D.$$

$\therefore$ P_1, Q, U, D are concyclic and this gives

$$\angle A'QD = \angle P_1UA' = \angle P_1UC = \angle B - \angle C \text{ from (2)}$$

From (1) we get $\angle A'QD = \angle B - \angle C = \angle TA'C$

$\therefore$ For the nine-point circle $A'T$ is a tangent and $A'D$ is a chord with $\angle A'QD = \angle TA'C$ implies that Q lies on the nine-point circle. Now Q is a common point to the incircle and the nine-point circle. The tangent at P_1 to the incircle is parallel to the tangent at A' to the nine-point circle. Therefore the tangent at Q to the nine-point circle

makes the same angle $\angle TA'Q$, with QP_1 and the tangent at Q to the incircle also makes the same angle with QP_1. This means that the incircle and the nine-point circle have the same tangent at Q or the two circles touch at Q.

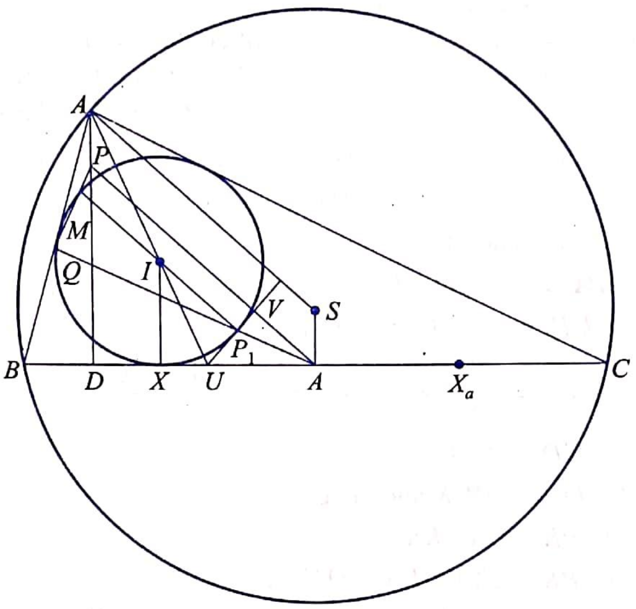


Fig. 4.60

Second Proof. As we have already seen in the first proof, we have $A'U \,.\, A'D = A'X^2$. UP_1 is a tangent to the incircle and UP_1, UX are symmetric with respect to AU. Also, AS and AD are symmetric with respect to AU. Therefore $UP_1 \perp AS$ and hence $UP_1 \perp A'P$, a diameter of the nine-point circle. $A'P_1$ meets the incircle at Q. Let $A'P$ and UP_1 meet at V. Then $\Delta A'DP \;|||\; \Delta A'VU$ and we have

$$A'V \cdot A'P = A'D \cdot A'U = A'X^2 = A'P_1 \cdot P_1Q.$$

This means that V, P, P_1 and Q are concyclic.

So, $\angle PQP_1 = \angle P_1VA' = 90°$ and $PQ \perp QP_1A'$. Hence Q lies on the nine-point circle for which $A'P$ is a diameter. The lines P_1IM and $A'VP$ are parallel and hence Q is

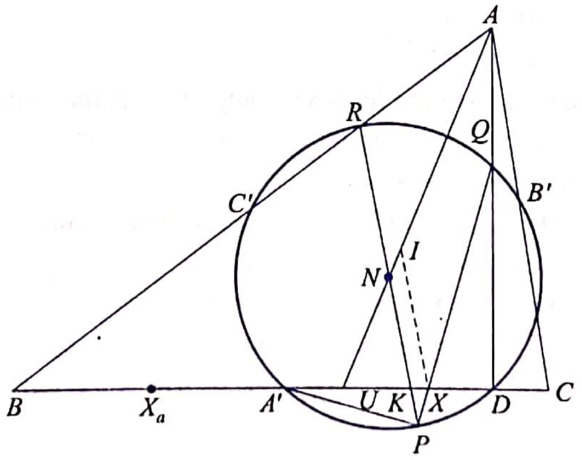


Fig. 4.61

collinear with the midpoints of P_1M and $A'P$. But P_1M and $A'P$ are diameters of the incircle and the nine-point circle; and Q lies on them implies that the two circles touch at Q.

Third Proof. Let PR be the diameter of the nine-point circle perpendicular to BC (Fig. 4.61). Let AI meet BC at U; X be the point of contact of the incircle with BC and let PX meet the nine-point circle again at Q. Let AD be the altitude through A. Now PR is the perpendicular bisector of DA' and hence arc DP = arc PA'. This implies that

$$\angle PA'D = (1/2)\ \angle A'C'D = (1/2)(\angle A'C'B - \angle DC'B)$$
$$= (1/2)(\angle A - (180° - 2\angle B))$$

(since $DA'B'C'$ is cyclic). Therefore,

$$\angle PA'D = (1/2)\ (\angle B - \angle C)$$

But $$\angle UIX = \angle UAD\ (\because\ IX \mid AD)$$
$$= \frac{\angle A}{2} - (90° - \angle B) = \frac{\angle B}{2} - \frac{\angle C}{2} \qquad (2)$$

From (1) and (2), $\angle PA'D = \angle UIX$.

This implies that $\Delta UIX \;|||\; \Delta PA'K$ and hence

$$XI \cdot PK = UX \cdot KA'$$

This gives $$2r \cdot PK = 2KA' \cdot UX = DA' \cdot UX \qquad (3)$$

But DU divides XX_a harmonically (as already seen in the earlier proofs) and A' is the midpoint of XX_a.

$\therefore$ $$A'U \cdot A'D = A'X^2 \text{ or } A'D\ (A'X - UX) = A'X^2,$$

Equivalently $$A'D \cdot UX = A'X\ (A'D - A'X) = A'X \cdot XD \qquad (4)$$

Substituting in (3), we get $2r \cdot PK = A'X \cdot XD = PX \cdot XQ$ (5)

Also $\angle XKR = 90° = \angle RQX$ implies that $XKRQ$ is concyclic.

$$PX \cdot XQ = PK \cdot KR \text{ and so } PX \cdot PQ = PK \cdot PR = PK \cdot 2p \qquad (6)$$

(where p is nine-point radius of ΔABC).

From (5) and (6) we get $\dfrac{r}{p} = \dfrac{XQ}{PQ}$.

This implies that $\Delta IXQ \;|||\; \Delta NPQ$. Therefore, Q, I, N are collinear and

$$\frac{XI}{IQ} = \frac{NP}{NQ} = 1.$$

This says that Q lies on the incircle as well; and this together with the fact that Q, N, I are collinear implies that the nine-point circle and the incircle touch at Q.

Fourth Proof. We have $SI^2 = R^2 - 2Rr$

$$IH^2 = 2r^2 - 2R\,r_p \text{ (where } r_p \text{ is the inradius of the orthic } \Delta DEF)$$
$$SH^2 = R^2 - 4Rr_p \text{ (Exercise)}$$
$$NI^2 = (1/2)\ (SI^2 + IH^2) - NH^2 \quad (\because\ IN \text{ is a median of } \Delta IHS)$$
$$= (1/2)(R^2 - 2Rr + 2r^2 - 2Rr_p) - (1/4)SH^2$$
$$= (1/2)\ (R^2 - 2Rr + 2r^2 - 2Rr_p - (1/2)R^2 + 2Rr_p)$$
$$= \left(\frac{R}{2} - r\right)^2$$

∴ The incircle and the nine-point circle touch each other. □

Note. 1. We have proved that the nine-point circle touches the incircle. In all the four proofs, some simple modifications and similar reasonings imply that the nine-point circle also touches each of the three escribed circles.

2. In the third proof, we actually give a construction of the point of contact of the incircle and the nine-point circle.

Theorem 60. (Pedal line Theorem). The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle.

Proof. Let A_1, B_1, C_1 be the feet of the perpendiculars from P a point on the circumcircle ABC on the sides BC, CA, AB respectively of ΔABC. P lies on the circles A_1BC_1, A_1B_1C, and AB_1C_1. Also PB, PC and PA are diameters of the circles A_1BC_1, A_1B_1C and AB_1C_1 respectively (Fig. 4.62). Therefore, $\angle C_1PA_1 = 180° - \angle B = \angle APC$. This gives

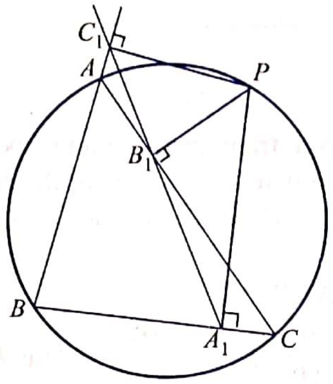


Fig. 4.62

$$\angle C_1PA = \angle C_1PA_1 - \angle APA_1 = \angle APC - \angle APA_1 = \angle A_1PC \qquad (1)$$

Also, quadrilateral A_1CPB_1 is cyclic implies that $\angle A_1PC = \angle A_1B_1C$ (2)

and quadrilateral AB_1PC_1 is cyclic gives $\angle C_1PA = \angle C_1B_1A$ (3)

(1), (2) and (3) imply that $\angle A_1B_1C = \angle C_1B_1A$ and hence A_1, B_1, C_1 are collinear.

Converse. If P is a point such that the feet of the perpendiculars from P onto the sides BC, CA, AB of ΔABC are collinear, then P must lie in the open region which is inside one of the angles of the triangle and P must be outside ΔABC.

Assuming that P is within $\angle B$ of ΔABC (Fig. 4.62), we retrace our steps in the above proof. A_1, B_1, C_1 are collinear implies that $\angle A_1B_1C = \angle C_1B_1A$.

Now AB_1PC_1 is cyclic implies that $\angle C_1B_1A = \angle C_1PA$ and the quadrilateral A_1CPB_1 is cyclic gives $\angle A_1B_1C = \angle A_1PC$. Therefore, we get $\angle A_1PC = \angle C_1PA$. Adding $\angle APA_1$ we get $\angle APC = \angle C_1PA_1$. Now BA_1PC_1 is cyclic implies that $\angle C_1PA_1 = 180° - \angle B$. Hence $\angle APC = 180° - \angle B$ or P lies on the circle ABC. □

Definition. If P_1, P_2, P_3 are the feet of the perpendiculars from a point P onto the sides of a triangle ABC, then $\Delta P_1P_2P_3$ is called the *pedal triangle* of P, with respect to ΔABC.

Theorem 60 says that if P lies on the circumcircle then the pedal triangle of P gets degenerated into a straight line also known as the *Simson line* of P.

Theorem 61. The sides of the pedal triangle of a point P with respect to the ΔABC are given by

$$P_2P_3 = a\left(\frac{AP}{2R}\right), P_3P_1 = b\left(\frac{BP}{2R}\right) \text{ and } P_1P_2 = c\left(\frac{CP}{2R}\right)$$

where R is the circumradius of ΔABC.

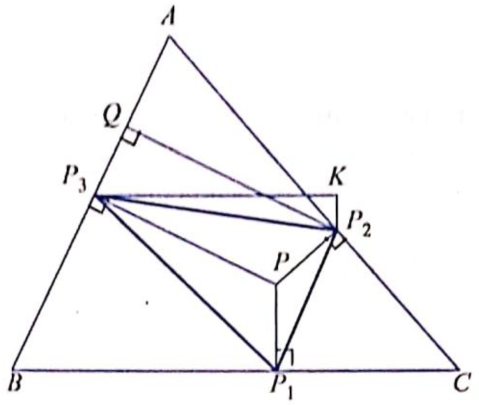


Fig. 4.63

Proof. Let P_1 P_2 P_3 be the pedal triangle of P with respect to ΔABC. Draw $P_2Q \perp AP_3$ (Fig. 4.63) and P_3K be the diameter through P_3 of the circle AP_3PP_2. Then $\angle P_3AP_2 = \angle P_3KP_2$ (angles in the same segment) and further $\angle AQP_2 = \angle KP_2P_3 = 90°$ ($\because$ P_3K is a diameter).

$\therefore$ $\Delta AQP_2 \,|||\, \Delta KP_2P_3$.

This gives $P_2P_3 = \frac{QP_2}{AP_2} . KP_3 = \frac{QP_2}{AP_2} . AP$ (as $KP_3 = AP$)

By the same argument applied to ΔABC we get

$$BC = \left(\frac{CF}{AC}\right) 2R$$

Also, from the similar triangles AQP_2 and AFC we get

$$\frac{QP_2}{AP_2} = \frac{FC}{AC} \text{ and thus } \frac{P_2P_3}{BC} = \frac{AP}{2R} \quad \text{or} \quad P_2P_3 = a\left(\frac{AP}{2R}\right)$$

Similarly, we get the other sides of the pedal triangle. ❑

Corollary 1. (Ptolemy's Theorem. See Theorem 22) If $ABCD$ is a cyclic quadrilateral then

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Proof. Take $D = P$ in the 'Pedal line Theorem', Then the pedal triangle of P degenerates into a straight line $A_1B_1C_1$. By the formula derived in Theorem II.

we have $$B_1C_1 = a\left(\frac{AP}{2R}\right), C_1A_1 = b\left(\frac{BP}{2R}\right), A_1B_1 = c\left(\frac{CP}{2R}\right).$$

Now $$A_1B_1 + B_1C_1 = A_1C_1 \text{ and hence } \frac{c \cdot CP + a \cdot AP}{2R} = b\left(\frac{BP}{2R}\right)$$

or $a . AP + c . CP = b . BP$.. In other words $BC . AD + AB . CD = AC - BD$ which is Ptolemy's theorem. ❑

Corollary 2. If ABC is a triangle and P is not on the arc CA of the circumcircle of ΔABC, then $AB . CP + BC . AP > AC . BP$.

Proof. By the converse of 'Pedal line Theorem', if P is not on the circumcircle, its pedal triangle $A_1B_1C_1$ is non-degenerate, but is a genuine triangle. Therefore $A_1B_1 + B_1C_1 > A_1C_1$ which gives

$$AB \cdot CP + BC \cdot AP > AC \cdot BP \text{ (See the proof of Corollary 1).}$$ ❑

Note. The Simson line of any vertex of a triangle is the altitude through that vertex and the Simson line of the point diametrically opposite to a vertex is the corresponding side. This is immediate from the definition of 'Simson lines'.

Theorem 62. If $A_1B_1C_1$ is the Simson line of a point P on the circumcircle of ΔABC, then the triangles PA_1B_1 and PBA are similar. Further,

1. $PA \,.\, PA_1 = PB \,.\, PB_1 = PC \,.\, PC_1$

2. $\dfrac{PA_1 \,.\, B_1C_1}{a} = \dfrac{PB_1 \,.\, C_1A_1}{b} = \dfrac{PC_1 \,.\, A_1B_1}{c}$

Proof. In Fig. 64, the quadrilateral PCA_1B_1 is cyclic.

$\therefore$ $$\angle A_1PB_1 = \angle A_1CB_1 = \angle BCA = \angle BPA.$$

Also, $$\frac{PA_1}{PB_1} = \frac{PB}{PA} \text{ (why?) and hence } \Delta PA_1B_1 \mathrel{|||} \Delta PBA.$$

This gives $PA \,.\, PA_1 = PB \,.\, PB_1$. Similarly $PB \,.\, PB_1 = PC \,.\, PC_1$.

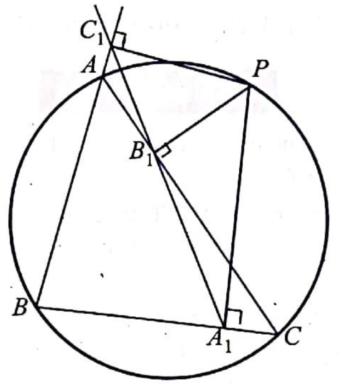


Fig. 4.64

This proves (1).

From Theorem 61, we have $B_1C_1 = a\left(\dfrac{AP}{2R}\right)$, $C_1A_1 = b\left(\dfrac{BP}{2R}\right)$ and $A_1B_1 = c\left(\dfrac{CP}{2R}\right)$.

Therefore, $\dfrac{PA_1 \,.\, B_1C_1}{a} = \dfrac{PA_1 \,.\, AP}{2R}$ and similarly

$$\frac{PB_1 \,.\, C_1A_1}{b} = \frac{PB_1 \,.\, PB}{2R}, \quad \frac{PC_1 \,.\, A_1B_1}{c} = \frac{PC_1 \,.\, PC}{2R}$$

Using (1), we get $$\frac{PA_1 \,.\, B_1C_1}{a} = \frac{PB_1 \,.\, C_1A_1}{b} = \frac{PC_1 \,.\, A_1B_1}{c}$$ ❑

Theorem 63. If P is a point on the circumcircle of a triangle ABC, then the Simson line of P bisects the line joining P and the orthocentre of the triangle.

Proof. Let $A_1B_1C_1$ be the Simson line of P and let the altitude BE meet the circumcircle again at T. Suppose that PB_1 meets the circumcircle again at K. Let $HL \parallel BK$ meet PK at L (Fig. 4.65). Now, $BHLK$ is a parallelogram and $PTBK$ is an isosceles trapezium (note that $PTBK$ is a cyclic quadrilateral). This means that $HL = BK$, $PT = BK$; also $HE = ET$ (Theorem 49, Chapter 4) and EA is the perpendicular bisector of HT.

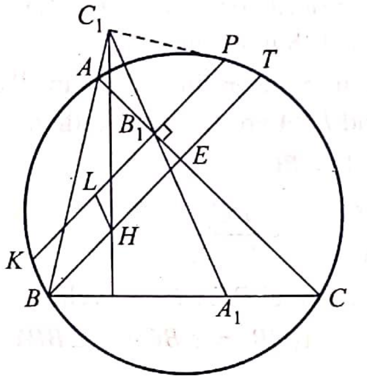


Fig. 4.65

$\therefore$ EA bisects perpendicularly the opposite side LP of the isosceles trapezium $LHTP$. This means that the Simson line $A_1B_1C_1$ passes through the midpoint B_1 of PL and it is parallel to LH. This implies that $A_1B_1C_1$ bisects PH. (See Fig. 4.65). ❑

EXERCISE 4.4

1. Prove that the angle which the external bisector of $\angle A$ of ΔABC makes with BC is half the difference of $\angle B$ and $\angle C$.
2. If the tangent at A to the circumcircle ABC meets BC at X then prove that $XA = XD = XD'$ where D, D' are the points where the internal and external bisectors of $\angle A$ meet BC respectively.
3. Let m_a, m_b, m_c be the medians through A, B, C of ΔABC. If XYZ is the triangle whose sides are of lengths m_a, m_b, m_c, prove that the medians of ΔXYZ are of lengths $3a/4$, $3b/4$, $3c/4$ respectively (where $a = BC$, $b = CA$, $c = AB$).
4. Let S be a given circle and G be a given point. How many triangles are there inscribed in S and having G as their centroid?
5. Show that a parallel to a side of ΔABC through its centroid G divides the area ΔABC in the ratio 4 : 5.
6. Show that in any ΔABC, $3s/2 < m_a + m_b + m_c < 2s$.
7. If X is the harmonic conjugate of the eentroid G of ΔABC with respect to A, D (where D is the midpoint of BC) show that $XD = AD$.
8. Find the locus of the centroid of a triangle on a fixed base and inscribed in a fixed circle.
9. If two point P, Q are equidistant from the centroid of a triangle ABC show that $PA^2 + PB^2 + PC^2 = QA^2 + QB^2 + QC^2$; and conversely.
10. ABC is a triangle. Find the locus of P if $PA^2 + PB^2 = PC^2$.
11. S is a given circle and O is a given point. If a variable chord AB subtends a right angle at O, find the locus of the midpoint of AB.
12. If X, Y, Z are the feet of the perpendiculars from the centroid G of ΔABC upon the sides BC, CA, AB prove that area $\Delta XYZ = 4\Delta^2(a^2 + b^2 + c^2)/9a^2b^2c^2$ where Δ = area of ΔABC.

13. Is the corollary to Theorem 29 true for external bisectors?

14. With notations as in Theorem 31, prove that any two points X, Y of $\{I, I_a, I_b, I_c\}$ are the extremities of a diameter of a circle passing through the other two vertices of ΔABC, not in line with X, Y.

15. Show that an external bisector of $\angle A$ of ΔABC is parallel to the line joining the points where the external (internal) bisectors of $\angle B$, $\angle C$ meet the circumcircle.

16. $DP = r$ is a given line segment. DQ is another line segment in line with PD such that D is in between P and Q: and further $DQ = r_a$. Let A be the harmonic conjugate of D with respect to P, Q. Then prove that AD is equal to the altitude h_a of ΔABC for which the inradius is r and exradius opposite to B is r_a.

17. Deduce from Exercise 16 that $h_a = rr_a/(r_a - r)$

$$h_b = rr_b/(r_b - r)$$
$$h_c = rr_c/(r_c - r).$$

18. In a variable triangle inscribed in a fixed circle and circumscribing a fixed circle prove that the sum of the exradii is constant.

19. Prove (a) $SI^2 + SI_a^2 + SI_b^2 + SI_c^2 = 12R^2$

(b) $II_a^2 + II_b^2 + II_c^2 = 8R(2R - r)$

(c) $I_aI_b^2 + I_bI_c^2 + I_cI_a^2 = 8R\ (4R + r)$.

20. Let ABC be a triangle with incentre I and circumcentre S. If XY is the diameter of the incircle perpendicular to the diameter PQ of the same circle (I, r) passing through S, show that the perimeter of $\Delta SXY = 2R$.

21. Let ABC be a triangle with X, Y, Z being the points of contact of the incircle with the sides of ΔABC. Show that the circles (A, AY), (B, BZ) and (C, CX) are tangent to each other.

22. ABC is a triangle; the excircle opposite to A touches BC at X_a. Prove that AX_a bisects the perimeter of the ΔABC.

23. XY is a straight line parallel to BC through the incentre I of ABC meeting AB, AC at X, Y respectively. Prove that $XY = XB + YC$.

24. ABC is a triangle; PQ, RS, TV are the tangents to the incircle parallel to BC, CA, AB respectively meeting AB, AC at P, Q; CA, CB at R, S; BC, BA at T, V respectively. Prove that the sum of the perimeters of ΔAPQ, ΔBTV and ΔCRS is equal to the perimeter of ΔABC.

25. With usual notations as in the text, prove that $AZ\ .\ BX\ .\ CY = r\Delta$.

26. ABC is a right triangle with $\angle A = 90°$. If the incircle of ΔABC touches BC at X, prove that, area $\Delta ABC = BX\ .\ XC$.

27. From any point inside a regular polygon perpendiculars are drawn to the sides of the polygon, prove that the sum of their lengths is a constant.

28. Prove that area of ΔABC = area of quadrilateral I_aYAZ.

29. If $\angle A = 60°$ in ΔBAC, prove that S, H, I, I_a, B, C all lie on a circle.

30. The incircle of ΔABC touches the sides BC, CA AB at X_1, Y_1, Z_1; the incircle of $\Delta X_1 Y_1 Z_1$ touches the sides of $\Delta X_1Y_1Z_1$ at X_2, Y_2, Z_2 and likewise points X_n, Y_n, Z_n are defined for $n > 2$. Prove that $\angle Y_n X_n Z_n = 60° + (-2)^{-n}\ (\angle A - 60°)$.

31. A variable straight line PQ cuts two fixed straight lines AB, CD at P, Q; the bisectors of $\angle AXY$ and $\angle CYX$ meet at X; find the locus of X.

32. In ΔABC, AD is the altitude through A; x, y, z are the inradii of ΔADC, ΔADB and ΔABC. Prove that $x^2 + y^2 = z^2$.

33. $ABCD$ is a quadrilateral circumscribing a circle. Prove that the incircles of ΔABC and ΔADC touch each other.

34. A circle of constant radius passes through a fixed point A and intersects two fixed straight lines AB, AC in B, C. Prove that the locus of the orthocentre of ΔABC is a circle.
35. Two rectangular chords, AB, CD of a circle revolve about a fixed point P. Show that the orthocentres of ΔABC, ΔABD describe the same circle.
36. Prove that the Euler line of ΔABC passes through A if and only if the triangle is either isosceles or right angled.
37. Let PQ be a diameter of the circumcircle of ΔABC whose centroid is G. Prove that PG bisects QH where H is the orthocentre of ΔABC.
38. ABC is a triangle: the tangents at A, B, C to the circumcircle form ΔPQR. Prove that the circumcentre of ΔPQR lies on the Euler line of ΔABC.
39. Prove that the circumcentre of ΔABC lies on the Euler line of ΔXYZ where X, Y, Z are the points of contact of the incircle with the sides of ΔABC.
40. If P, Q, R are the midpoints of AH, BH, CH show that $\Delta PQR \equiv \Delta A'B'C'$ (medial traingle) where H is the orthocentre of ΔABC.
41. Prove that $\Delta DB'C'$ is congruent to ΔPQR (notations as in 40; D is the foot of the altitude from A).
42. With notations as in 40, prove that SP is bisected by AA' where S is the circumcentre.
43. If a variable triangle has a fixed base and a constant circumradius, show that its nine-point circle is tangent to a fixed circle.
44. A variable ΔABC has its vertex A fixed and has a fixed nine-point circle; prove that the locus of its orthocentre is a fixed circle.
45. If the pedal line of P with respect to ΔABC is parallel to AS, prove that $PA \parallel BC$.
46. Let the altitude AD of ΔABC meet the circumcircle of ΔABC at D'. Prove that the pedal line of D' is parallel to the tangent at A to the circle.
47. Prove that the feet of the perpendiculars from the midpoint A' of BC to the sides of the orthic triangle are collinear.
48. P is a point on the circle ABC and H is the orthocentre of ΔABC, prove that the pedal line of P bisects PH.
49. The perpendiculars from P, a point on circle ABC, to BC, CA, AB meet the circle again at X, Y, Z. Prove that $\Delta ABC \equiv \Delta XYZ$.
50. Let X be any point on the circumcircle of ΔABC. Let the altitudes AD, BE, CF meet the circle again at D', E', F'; if XA', XB', XC' cut BC, CA, AB at P, Q, R, prove that P, Q, R are collinear with the orthocentre H of ΔABC and parallel to the pedal line of X.

4.5 CONSTRUCTIONS

Let us begin with some very basic constructions.

Construction 1. From a given a point in a given straight line, construct a straight line, making with the given line an angle equal to a given angle.

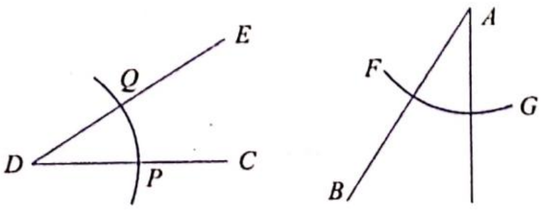


Fig. 4.66

Let A be the given point, AB the given straight line and $\angle CDE$ be the given angle. With D as centre and any convenient radius, draw an arc of the circle meeting DC at P and DE at Q. With A as centre and same radius draw an arc Γ of a circle meeting AB at F. With centre F and radius equal to PQ draw an arc meeting the arc Γ at G. Then AG is the required straight line. (Fig. 4.66).

Proof. By construction, the three sides of ΔAFG are equal in length to the corresponding sides of ΔDPQ. Hence the two triangles are congruent. $\angle FAG = \angle PDQ = \angle CDE$ and AG is the required line. ❑

Construction 2. Let $\angle BAC$ be a given angle. Bisect the angle $\angle BAC$. With centre A and any radius draw an arc of a circle meeting BA at D and AC at E. With D, E as centres and suitable radius draw two arcs meeting at F. Then AF bisects the given angle $\angle BAC$ (Fig. 4.67).

Proof. By construction, $DF = EF$ and hence the three sides of ΔADF are equal to the corresponding sides of ΔAEF, $\therefore$ $\Delta ADF \equiv \Delta AEF$ and this means that $\angle DAF = \angle EAF$ or AF bisects $\angle BAC$. ❑

Construction 3. Given a line segment AB, bisect AB.

Choose a suitable radius and draw arcs of circles with centres A and B to meet at X and Y. Let XY meet AB at C. Then C bisects AB. (Fig. 4.68).

Proof. By construction, $AX = BX = AY = BY$. Therefore $\Delta AXY \equiv \Delta BXY$. Now $\angle AXY = \angle BXY$ implies $\Delta AXC = \Delta BXC$ and so $AC = BC$. ❑

Construction 4. To draw the perpendicular to a given straight line from a given point on it. C is the given point on a given line AB. Cut off equal lengths CD, CE on AB as in Fig. 4.69. With any radius bigger than CD, draw arcs with centres D and E meeting at X. Then $CX \perp AB$.

Proof. By construction $DX = EX$, $DC = CE$ and hence $\Delta DCX = \Delta ECX$.

Therefore $\angle DCX = \angle ECX = 90°$ ❑

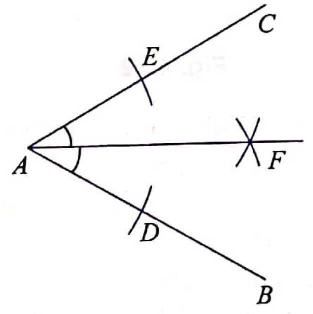


Fig. 4.67

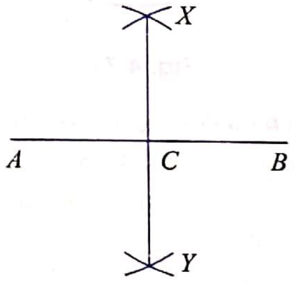


Fig. 4.68

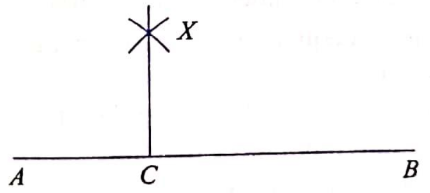


Fig. 4.69

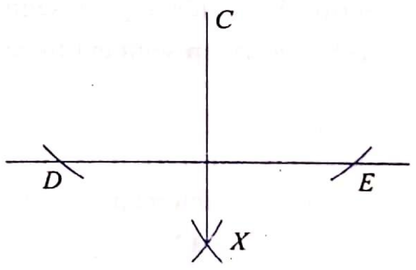


Fig. 4.70

Construction 5. To draw the perpendicular to a given straight line from a given point not on it.

C be the given point and AB the given straight line. With C as centre draw a convenient arc of a circle meeting AB at D, E. With suitable radius, draw arcs of circles with centres D and E meeting at X (Fig. 4.70). Then $CX \perp AB$.

Proof. By construction, $DX = EX$, $CD = CE$. So $\Delta CDX \equiv \Delta CEX$ and $\angle DCX = \angle ECX$. Therefore $\Delta CDG \equiv \Delta CEG$. Hence $\angle CGD = \angle CGE = 90°$ or $CX \perp AB$. ❑

Construction 6. Draw a circle passing through three non-collinear points.

Using the earlier constructions, draw the perpendicular bisectors of AB, AC to meet at S. Then the circle with S as centre and radius SA is the required circle.

Proof. S is on the perpendicular bisector of AB and AC implies that $SA = SB$ and $SA = SC$. Thus $SA = SB = SC$ or the circle with centre S and radius SA passes through A, B and C. ❑

Construction 7. To draw a straight line parallel to a given straight line through a given point.

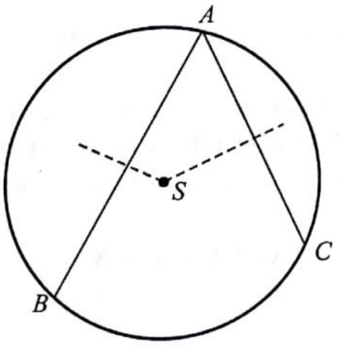


Fig. 4.71

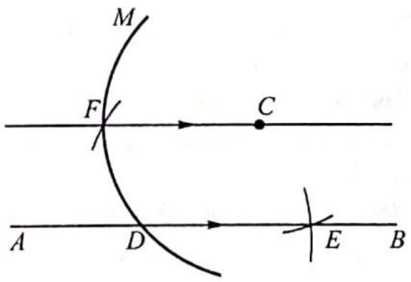


Fig. 4.72

Let AB be the given straight line and C the given point. With C as centre draw an arc Γ meeting AB at D. With the same radius draw an arc with D as centre to meet AB at E. Now with D as centre and radius equal to CE, draw an arc of a circle cutting Γ at F. Join CF and CF is the required straight line parallel to AB.

Proof. By construction $CF = CD = DE$. Therefore $\Delta CDF \equiv \Delta DCE$. This means that $\angle FCD = \angle EDC$. These are alternate angles for the transversal CD cutting AB and CF. Therefore $AB \parallel CF$. ❑

Construction 8. Divide a given segment into a given number of equal parts.

Let AB be the given segment to be divided into n equal parts. Draw through A any convenient ray AC. Along AC cut off equal segments AA'_1, $A'_1A'_2$, $A'_2A'_3$, ... $A'_{n-1}A'_n$. Draw straight lines through A'_1, A'_2 ..., A'_{n-1} parallel to A'_nB meeting AB at A_1, A_2 ..., A_{n-1} respectively. Then A_1, A_2..., A_{n-1} are the required points of division.

Proof. By the equal intercepts theorem on parallel lines, we have ❑

$$AA_1 = A_1A_2 = A_2A_3 = \ldots = A_{n-1}B.$$

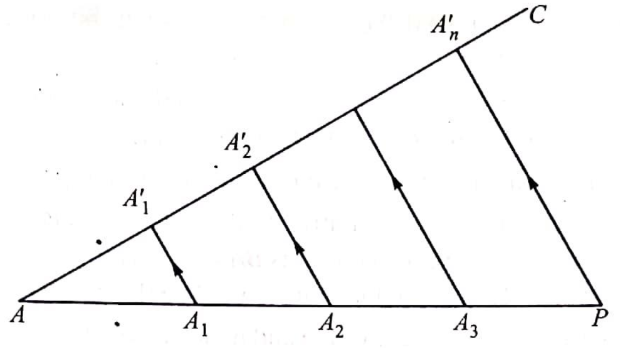


Fig. 4.73

Construction 9. To construct *the fourth proportional* to three given segments. Given three segments of lengths x, y, z we wish to find a segment AG such that $x : y = z : AG$.

Draw any convenient angle $\angle BAC$. On AB cut off AD, AE equal to x, y respectively. On AC cut off $AF = z$. See Fig. 4.74. Draw $EG \parallel DF$ meeting AC at G. Then $\Delta ADF \;|||\; \Delta AEG$ and hence

$$\frac{AD}{AE} = \frac{AF}{AG} \quad \text{or} \quad \frac{x}{y} = \frac{z}{AG}$$

Thus AG is the required fourth proportional to x, y, z. ❑

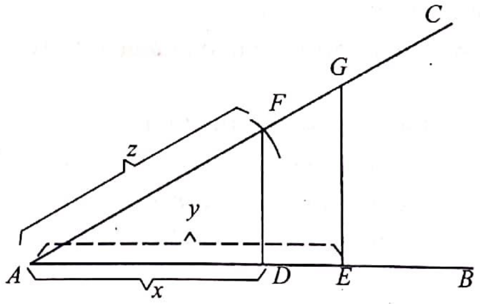
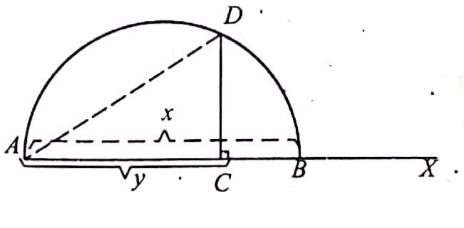


Fig. 4.74

Construction 10. To construct *the mean proportional* to two given segments. Let x and y be the lengths of the two given segments. It is required to find a segment of length z such that $x/z = z/y$ or $z^2 = xy$.



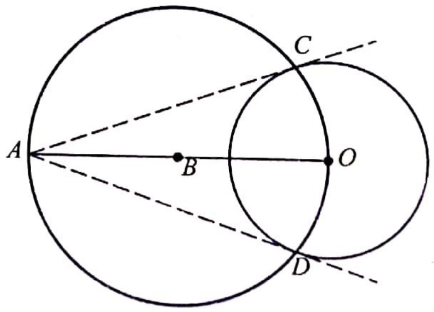


Fig. 4.75 Fig. 4.76

Draw a straight line AX. On AX, cut off AB equal to x and AC equal to y. (Fig. 4.75). In our figure we have taken $y < x$. On AB draw the semicircle and draw $CD \perp AB$

meeting this semicircle at D. (All these constructions can be done using our earlier constructions). $AD = z$ is the required mean proportional.

Proof. By construction, D lies on the semicircle on AB and hence $\angle ADB = 90°$. And ΔDAB ||| ΔCAD and hence $AB/AD = AD/AC$ or $x/z = z/y$. ❑

Construction 11. Draw a tangent to a circle from an external point.

Let A be the given point and O the centre of the given circle. Bisect AO and find the midpoint B of AO. With B as centre and radius $BA = BO$ draw a circle cutting the given circle at C, D. Then AC, AD are the tangents from A to the given circle. (Fig. 4.76).

Proof. By construction ACO is a semicircle and hence $\angle ACO = 90°$. Similarly $\angle ADO = 90°$. Hence AC, AD are the tangents from A. ❑

Construction 12. To draw a direct common tangent to two given circles.

Let A, B be the centres of the two given circles with radii, r_1, r_2 respectively. We first assume that $r_1, > r_2$. Draw the circle with centre A and radius $r_1 - r_2$. Draw a tangent BC to this circle [Fig. 4.77*a*]. Let AC cut the circle (A, r_1) at D. Draw $BE \parallel AD$ cutting the circle (B, r_2) at E. Then DE is a direct common tangent to the two given circles.

Proof. We have $CD = r_1 - (r_1 - r_2) = r_2 = BE$. By construction $CD \parallel BE$ and hence $CBED$ is a parallelogram. BC is a tangent to $(A, r_1 - r_2)$ implies that $BC \perp AC$. Therefore $CBED$ is a rectangle and hence $\angle CDE = \angle DEB = 90°$. Therefore DE is a common tangent to the two circles. When $r_1 = r_2$, construct the rectangle $ABED$ on AB to get DE as a direct common tangent [Fig. 4.77*b*]. ❑

Construction 13. Draw a transverse common tangent to two given non-intersecting circles.

We imitate what we did in the last construction.

With A as centre draw the circle of radius $r_1 + r_2$. Draw the tangent BC to this bigger circle (Fig. 4.78). Let AC cut the circle (A, r_1) at D. Draw $BE \parallel DA$ meeting the circle (B, r_2) at E. DE is a transverse common tangent.

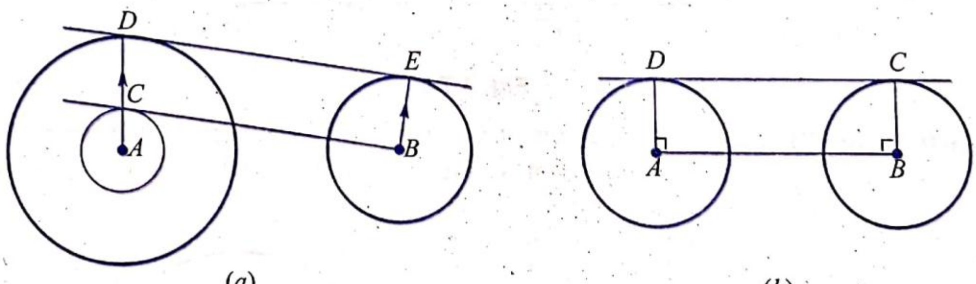


Fig. 4.77

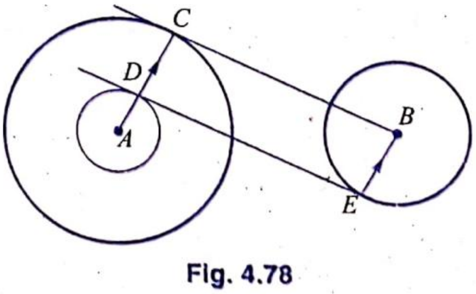


Fig. 4.78

Proof. Exercise (similar to the previous construction). ❑

Construction 14. Through a given point outside a given circle draw a secant so that the chord determined by it subtends an angle at the centre equal to the acute angle between the secant and the diameter through the given point.

Let O be the centre of the given circle and A be the given point outside the circle. Draw the circle with centre A and radius AO meeting the given circle at Q, Q' (Fig. 4.79). Join AQ. Then AQ is a secant satisfying our requirements.

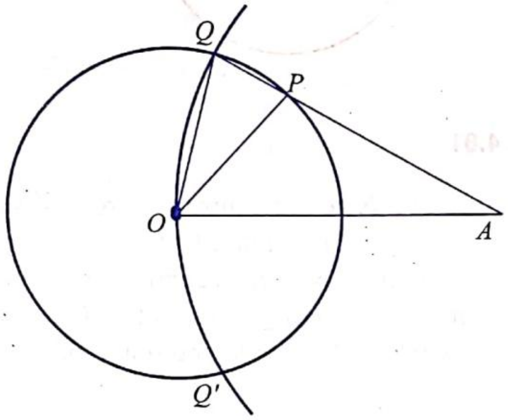


Fig. 4.79

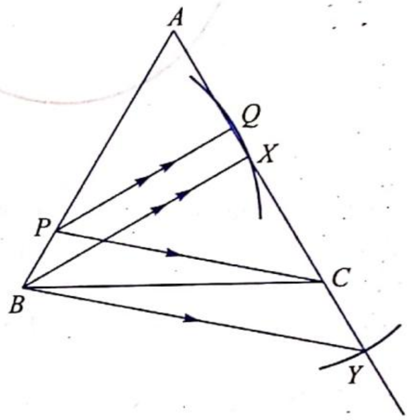


Fig. 4.80

Proof. Let AQ meet the circle again at P. We have $OQ = OP$ and $AQ = AO$ (by construction) and $\angle OQP = \angle OQA$. Therefore the vertical angles in the isosceles triangles QOP and QOA are also equal.

So, $\angle QOP = \angle OAQ$ and AQ is a secant satisfying our requirements. ❑

Question. Can you replace the adjective 'acute' by 'obtuse' in the above problem? Can you draw a secant AQ so that $\angle QOP$ is the obtuse angle between AQ and AO?

Remark. The above problem has two solutions symmetrical about AO.

Construction 15. Let ABC be a triangle. Find two points P, Q on AB, AC,. produced if necessary such that the line segments AP, PQ, QC are all equal.

Draw the circle with radius BA and centre B to cut AC at X. Draw the circle with centre X and radius BA to cut AC at Y. Draw the parallel CP to YB meeting AB at P (Fig. 4.80). Let the parallel to BX through P meet AC at Q. Then P, Q are the required points.

Proof. By construction $\Delta PQC \parallel\!\mid \Delta BXY$. Again by construction $XB = XY$. So, ΔPQC is also isosceles and $PQ = QC$. The triangles APQ and ABX are similar and $AB = BX$ by construction. Therefore $AP = PQ$. Thus $AP = PQ = QC$.

Remark

1. The point X is uniquely fixed on AC. We have two symmetric positions for Y on AC and we get two solutions.
2. What happens if $\angle A = 90°$ in the above construction?

Construction 16. Let S, S' be two given circles. Find points P, Q on S, S' respectively such that PQ is equal to a given length and PQ is parallel to a given direction.

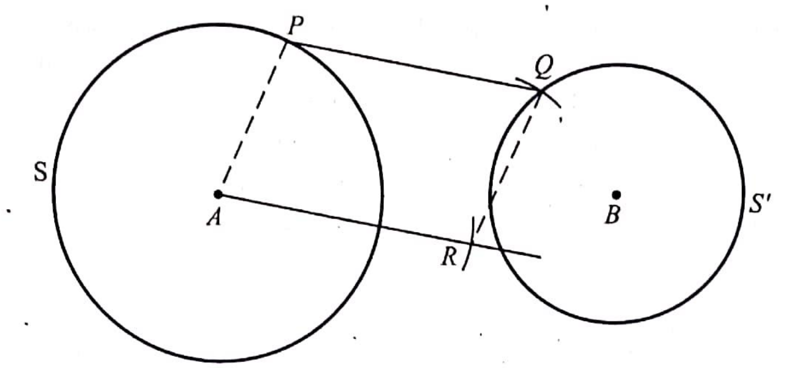


Fig. 4.81

Let A be the centre of circle S with radius r and B be the centre of circle S' with radius r'. Draw the line through A parallel to the given direction l. Cut off AR on this straight line such that AR is equal to the given length m. With R as centre and radius r draw a circle meeting S' at Q. Draw the straight line through Q parallel to AR and lay off $QP = QR$ so as to form a parallelogram in which QR is a side and not a diagonal. The points P, Q are the required points.

Proof. $PQ \parallel l$ and $|PQ| = m$ are true by construction. The only thing remains to be proved is that P is on the circle S. In the parallelogram $ARQP$ we have $AP = RQ$. But RQ = radius of S by construction. Hence, P lies on S. ❑

Note. We have two positions of R on either side of A. The circle with each one of these points as centre and radius equal to that of S(1) may cut S' at two points (2) may touch S' (3) may not cut S' at all. Therefore, the problem has four solutions, three solutions, two, one or no solutions.

Construction 17. Draw a circle passing through two given points and subtending a given angle at a third given point.

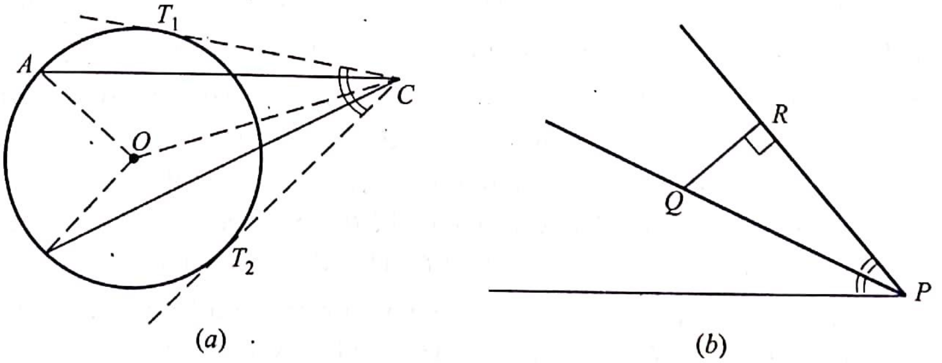


Fig. 4.82

It is required to find a circle through the two given points A, B and subtending a given angle at C; *i.e.*, if CT_1, CT_2 are the tangents to the circle then $\angle T_1CT_2$ is the given angle, $\angle P$. See Fig. 4.82(a). Take any point Q on the internal bisector of $\angle P$ and draw QR perpendicular to one of the sides of the angle [Fig. 4.82(b)]. Divide AC internally and externally in the ratio $QR : QC$ to get the points E, F. Similarly divide BC in the ratio $QR : QC$ internally and externally at G, H. Let the circles on EF and GH as diameters meet at O. Then O is the centre of the required circle.

Proof. By construction $OA/OC = QR/QC = OB/OC$. If CT_1, CT_2 are the tangents to the circle from C then we have

$$\frac{OT_1}{OC} = \frac{OA}{OC} = \frac{QR}{QC} \quad \text{(since } OA = OT_1\text{)}.$$

Now in the right angled triangles OT_1C and QRC we have $OT_1/QR = OC/QC$ and hence the two triangles are similar. Therefore $\angle OCT_1 = \angle QCR = \angle P/2$. Hence $\angle T_1CT_2 = \angle P$ = given angle. ❑

Construction 18. Through two given points on a circle, draw two parallel chords whose sum will have a given length.

Suppose AD, BC are the two required chords passing through the two given points A and B on the circle S with centre O. Then the trapezium $BCDA$ being cyclic, is isosceles. Therefore $CD = BA$. Now CD and BA are chords of the same length. So, CD touches the circle with centre O and radius OE (E is the midpoint of AB); and the point of contact is F the midpoint of CD. Also $2EF = AD + BC = 2s$ (given length). Thus we have the following construction. Find the midpoint E of AB and draw the circle with centre O and radius OE; also draw the circle with centre E and radius s (where $AD + BC = 2s$ = the given length) to cut the circle (O, OE) at F. Let the tangent to the circle (O, OE) at F meet the given circle S at C, D (Fig. 4.83). Then AD, BC are the required chords.

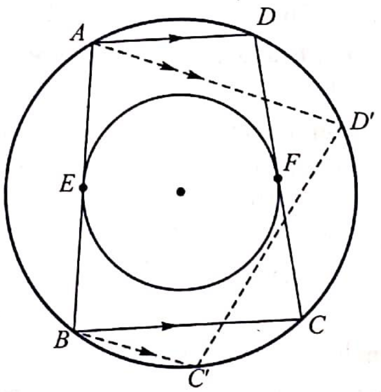


Fig. 4.83

Proof. Since by construction, the chords AB and CD are equidistant from the centre, we have $AB = CD$. Therefore $ABCD$ becomes an isosceles trapezium, and $AD + BC = 2EF = 2s$ = given length. ❑

Note. If $s > 2OE$ then the circle with centre E and radius s will not cut the circle (O, OE). If $s < 2OE$, we get two points of intersection F, P' giving rise to two solutions.

Construction 19. Given a line segment AB and an angle α, construct a segment of a circle such that AB subtends angle α at any point on this circular arc.

Draw AT such that $\angle BAT = \alpha$ the given angle. Let the perpendicular bisector of AB and the perpendicular to AT at A meet at O. Then the arc Γ of the circle with centre O and radius OA is the required arc (Fig. 4.84).

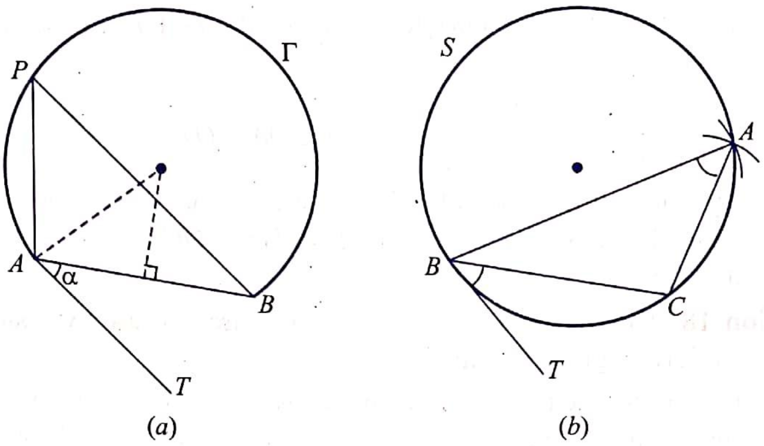


Fig. 4.84

Proof. If P is any point on this arc (Fig. 4.84) we have $\angle APB = \angle BAT = \alpha$ (by construction AT is a tangent to the circle (O, OA) at A). ❑

Construction 20. Given the base BC, the vertical angle A and the side AC, construct ΔABC.

As in construction 19, draw the segment S of the circle on BC such that BC subtends $\angle A$ at every point on this arc. With C as centre and radius AC draw an arc cutting the circular arc S at A. [Fig. 4.84(b)]. Then it is clear that ΔABC satisfies our requirements. ❑

Construction 21. Construct a triangle given the base, the opposite angle and the difference of the other two sides.

Let ABC be the required triangle. Suppose we are given a, A and $b - c$ Take D on AC such that $AD = AB$. This makes $CD = b - c$. Then ΔADB is isosceles and $\angle ADB = \angle ABD = 90° - A/2$. See Fig. 4.85. Therefore the exterior angle $BDC = 90° + A/2$. Now in ΔBDC, we know the base $BC = a$, $CD = b - c$ and $\angle BDC = 90° + A/2$. Therefore we may construct ΔBDC. Third vertex A lies on CD produced and on the perpendicular bisector of BD. ❑

Note. If $a > b - c$, the problem has no solution.

Construction 22. Construct a triangle given the base, the difference of the other two sides and the altitude to one of these sides.

Suppose we are given $BC = a$, $b - c$ and the altitude $CF = hc$.

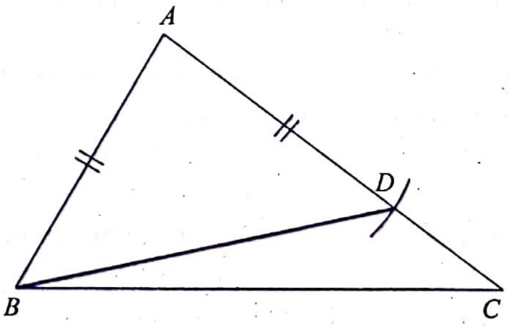


Fig. 4.85

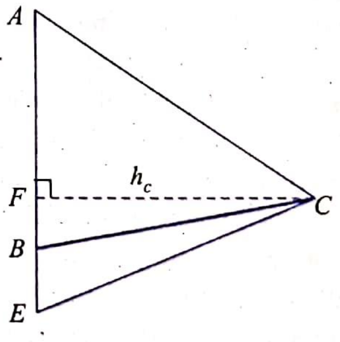


Fig. 4.86

Let ABC be a triangle with the required properties. Let E be a point on AB produced such that $BE = b - c$ or $AE = AC$. In ΔBEC, we know BE, BC and the altitude through C (Fig. 4.86). Therefore the ΔBEC may be constructed. (Note that C must be at a distance h_c from BE and hence lies on the straight line parallel to BE at a distance h_c, from BE). Once ΔBEC is known, we can fix the third vertex A on EB.

Construction 23. Construct ΔABC given a, A and $b + c$ (with the usual notations).

Let ABC be a triangle satisfying our requirements. Produce BA to D such that $AD = AC$. Now, ΔACD is isosceles and $\angle ADC = \angle ACD = (1/2)\angle BAC = A/2$. In ΔBCD, we know $BC = a$, $BD = b + c$ and $\angle BDC = A/2$. Therefore we can construct ΔBDC. A lies on the perpendicular bisector of CD and on BD. ❑

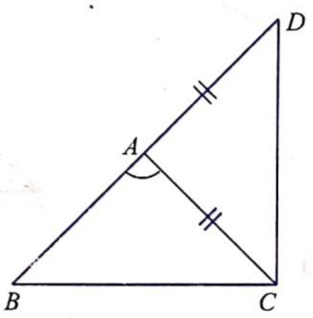


Fig. 4.87

Note. If $a > b + c$, the problem has no solution. Prove that the problem has two, one or no solutions.

Construction 24. Construct a triangle given its perimeter, the angle opposite the base and the altitude to the base.

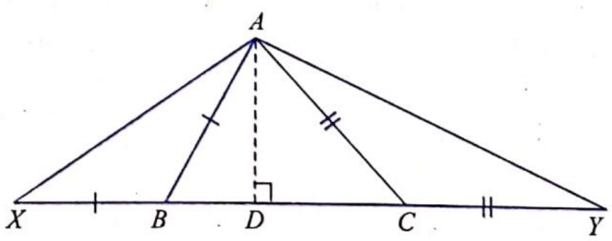


Fig. 4.88

Suppose we are given $2s$, A, h_a (usual notations). Let ABC be the required triangle. Produce BC either way and let $BX = BA$ and $CY = CA$. Then $XY = XB + BC + CY = a + b + c = 2s$.

$$\Delta XAB \text{ and } \Delta YAC \text{ are isosceles triangles.}$$

Therefore $\angle AXB = \angle XAB = (1/2)\angle ABC$ and similarly $\angle AYC = (1/2)\angle ACB$.

Therefore $\angle XAY = (1/2)\angle B + \angle A + (1/2)\angle C = 90° + \angle A/2$.

Now ΔXAY may be constructed since we know the base $XY = 2s$, the vertical angle $\angle XAY = 90° + \angle A/2$ and the altitude h_a through A. Also $BA = BX$ and $CA = CY$ imply that B and C lie on the perpendicular bisectors of AX and AY respectively.

Thus ΔABC may be constructed. ❑

Note. This problem may have two solutions symmetric about the perpendicular bisector of XY or just one solution or none at all.

Construction 25. Given a, A, $h_b + h_c$ (with the usual notations) construct the triangle ABC.

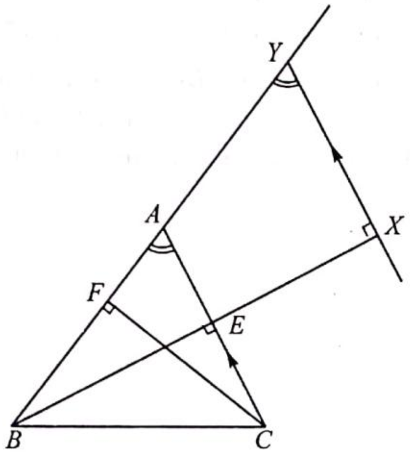


Fig. 4.89

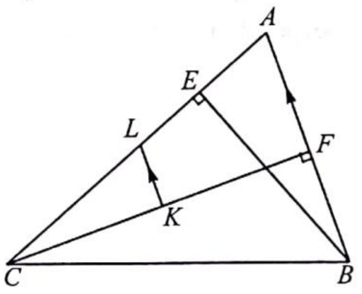


Fig. 4.90

Let ABC be a triangle meeting our requirements. Produce BE to X such that $EX = h_c = CF$. Draw XY parallel to CA meeting BA at Y. Then we have

$$\angle BYX = \angle BAC = \angle A \quad \text{(corresponding angles).}$$

$\angle BXY = \angle BEA = 90°$. Now, in the right angled triangle BXY we know $BX = h_b + h_c$ and the acute angle $\angle BYX$. Therefore ΔBXY may be constructed. We observe that if $AZ \parallel BE$ meeting XY at Z (Fig. 4.89) then $AZXE$ is a rectangle and $AZ = EX$. Therefore $AZ = EX = h_c$, In the right angled triangles ACF and AYZ, we have $AZ = EX$. Therefore $AZ = EX = h_c$. In the right angled triangles ACF and AYZ, we have $AZ = CF$, $\angle AYZ = \angle CAF = \angle A$.

Therefore $\Delta ACF \equiv \Delta AYZ$; $\therefore$ $AY = AC = b$ and hence

$$BY = BA + AY = c + b.$$

In the isosceles triangle AYC, $\angle AYC = \angle ACY = \angle A/2$. Also $\angle A = \angle AYX$ implies that YC bisects $\angle Y$. Therefore C is the intersection of this angular bisector of $\angle AYX$ and the circle (B, a). Once C is fixed, A is determined on BY by the perpendicular bisector of YC. ❑

Construction 26. Given a, A, $h_c - h_b$, construct ΔABC. Again let ABC be the triangle satisfying our requirements.

Let BE, CF be the altitudes through B and C respectively. Let K be the point on CF such that $KF = BE$. This gives $KC = h_c - h_b$. Let $KL \parallel AB$ meeting AC at L (Fig. 4.90). Now the right triangle CKL may be constructed since we know CK and $\angle KLC = \angle A$. The triangle ABL is isosceles (why?) and therefore $\angle ALB = 90° - A/2$. Also $\angle ALK = 180° - A$ (Fig. 4.90). Therefore LB is the bisector of $\angle ALK$. Hence B lies on the angular-bisector of $\angle ALK$ and the circle (C, a). This determines B. To fix A, we note that A is the intersection of CL and the perpendicular bisector of BL. ❑

Construction 27. Find points, D, E on AB and AC of ΔABC so that $BD = DE = EC$.

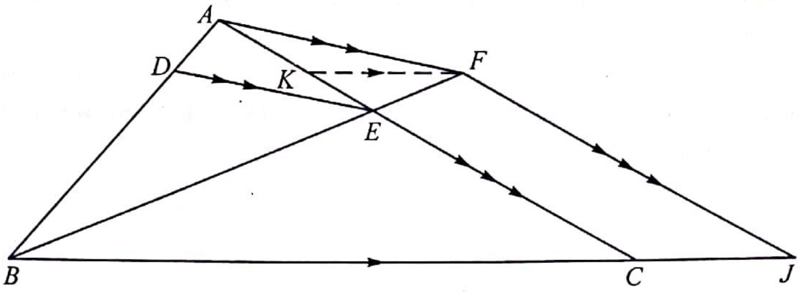


Fig. 4.91

Suppose D, E are the required points (Fig. 4.91). Draw $AF \parallel DE$ meeting BE at F and $FJ \parallel AC$ meeting BC at J. From the similar triangles BDE and BAF we get $DE/AF = BD/BA = BE/BF$. But $BD = DE$ and hence $AF = AB$, Again from the similar triangles BEC and BFJ we get $EC/FJ = BE/BF = BC/BJ$; and therefore $DE/AF = BE/BF = EC/FJ$. But $DE = EC$ and hence $AF = FJ$. Thus $BA = AF = FJ$. The quadrilateral $BAFJ$ is constructed as follows. Take K on CA such that $CK = AB$. Draw the parallel through K to BC meeting the circle with centre A and radius AB at F. Let the parallel to AC through F meet BC at J. This completes the construction of the quadrilateral $BAFJ$. AC and BF determines E. The parallel to AF through E determines D. ❑

Construction 28. Given A, $a + b$, $a + c$ construct ΔABC.

Let ΔABC be the required triangle. Produce AB and AC and take points D, E on AB, AC such that $BD = BC = CE$ (Fig. 4.92). This gives $AD = c + a$, $AE = b + a$. This means that ΔADE can be constructed, as we know AD, AE and the included angle $\angle A$. Now find points B, C on the sides AD, AE of ΔADE such that $DB = BC = CE$ using the previous construction. ❑

Construction 29. Construct ΔABC given the perimeter $2s$ and two angles B and C.

See Fig. 4.93. Draw $XY = 2s$ and $\angle YXK = \angle B$ and $\angle XYL = \angle C$. Bisect the angles KXY and LYX and let the bisectors meet at A. Draw $AB \parallel KX$ and $AC \parallel LY$ meeting XY at B, C respectively. Then ΔABC is the required triangle.

Proof. By construction $\angle AXB = \angle B/2$ and $\angle XBA = 180° - B$: $\angle BAX = \angle B/2$ which means that $XB = AB$ and similarly $AC = CY$. Thus $AB + BC + CA = XY = 2s$. ❑

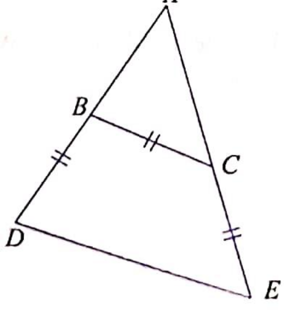


Fig. 4.92

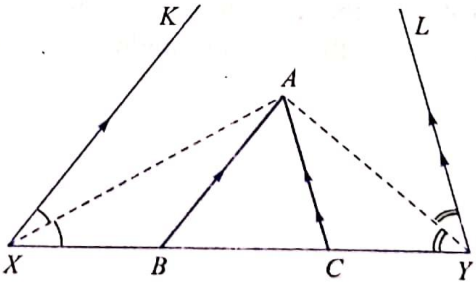


Fig. 4.93

Construction 30. Let $\angle XOY$ be a given angle and A a given point. Draw a straight line through A such that the segment intercepted on it by the sides of the given angle is divided by A in a given ratio $\lambda : \mu$.

Take any point C on OY. Find D on AC such that $AC : AD = \lambda : \mu$. Draw $DP \parallel YO$ meeting OX at P. Then AP is the required line (Fig. 4.94). The triangles AQC and APD are similar. Therefore $AQ/AP = AC/AD = \lambda/\mu$. ❑

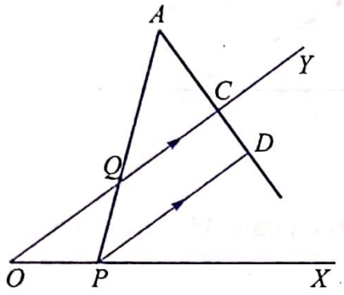


Fig. 4.94

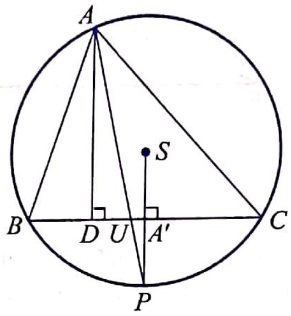


Fig. 4.95

Construction 31. Construct ΔABC given a, A, $A'D/A'U$ where D is the foot of the altitude from A and U is the point where the internal bisector of $\angle A$ meets BC and A' is the midpoint of BC.

Let ABC be a triangle with the required properties and S be the circumcentre of ΔABC. SA' meets the circumcircle at P. Then AU also meets the circumcircle at P (Why?). P bisects the are BC. (Fig. 4.95) $\Delta ADU \,|||\, \Delta PA'U$ and hence $A'U/UD = A'P/AD$. Now, BC and $\angle A$ are given and therefore the circumcircle of ΔABC can be constructed. So, $A'P$ is determined. Also $A'U/UD$ is known from the hypothesis. Therefore $A'U/UD = A'P/AD$ gives $AD \cdot \Delta ABC$ is readily constructed now. ❑

Construction 32. Construct ΔABC given R, $b + c$, $\angle B - \angle C$.

Let ΔABC be the required triangle. Let AP be parallel to BC meeting the circumcircle again at P. (Fig. 4.96). Now $APCB$ is an isosceles trapezium. $\angle ABP = \angle ABC - \angle PBC = \angle B - \angle BPA = \angle B - \angle BCA = \angle B - \angle C$. The circumradius R and $\angle ABP = \angle B - \angle C$ determine AP (Why?).

Now in ΔABP, we know base AP, the vertical angle $\angle ABP = \angle B - \angle C$ and the sum $AB + BP = AB + AC = b + c$. Hence we may construct ΔABP. The straight line through B parallel to AP meets the circumcircle at the vertex C of ΔABC. ❑

Construction 33. Construct a triangle given the median, the altitude and the bisector from the same vertex (Given h_a, m_a, t_a)

Let AD be the altitude h_a, AU the bisector t_a and AA' the median m_a. We can construct the right angled triangles ADU and ADA' using the given data. (Fig. 4.97).

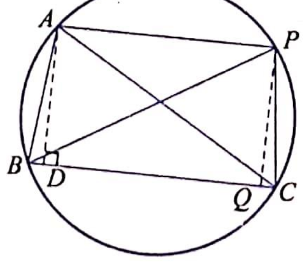


Fig. 4.96

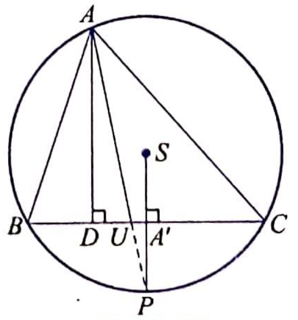


Fig. 4.97

The circumcentre S lies on the perpendicular to DA' through A' such that $\angle UAS = \angle DAU$. (why?). Once S is determined draw the circle with centre S and radius SA to cut DA' at B and C. ABC is the required triangle. ❑

Construction 34. Construct ΔABC given the medians m_a, m_b and m_c.

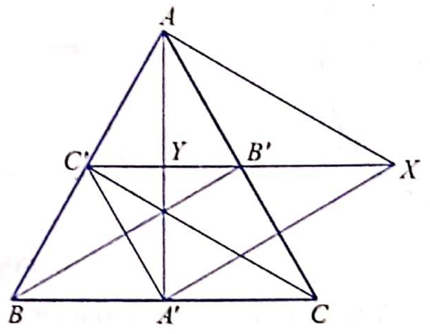


Fig. 4.98

Suppose ABC is the required triangle with A', B', C' as the midpoints of the sides BC, CA, AB respectively. Produce $C'B'$ to X such that $B'X = C'B'$. Then in the quadrilateral $AXCC'$, the diagonals AC and $C'X$ bisect each other. Therefore $AXCC'$ is a parallelogram. This gives $AX = CC' = m_c$. From the parallelogram $BA'XB'$ we get $A'X = BB' = m_b$. This means that the sides AX, XA', $A'A$ of the triangle AXA' are m_c, m_a, m_b respectively, and can be constructed. To go from $\Delta AXA'$ to ΔABC, we observe that XY is a median of $\Delta AXA'$ (See Fig. 4.98). Therefore the point Y can be determined on AA' and further B', C' satisfy $B'Y = YC' = (1/3)XY$. This means that B', C' can be located on XY. From $\Delta AB'C$, the ΔABC is readily constructed.

Construction 35. Given the altitude h_a, a and the bisector t_a, construct ΔABC.

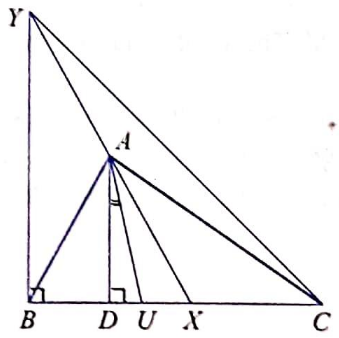


Fig. 4.99

Suppose ΔABC satisfies our requirements. See Fig. 4.99. Let X be a point on DC such that $BD = DX$. Then $\angle XAC = \angle AXD - \angle ACB = \angle B - \angle C$ (since ΔABX is isosceles). As we have seen earlier $\angle DAU = (1/2)(\angle B - \angle C)$ (Theorem 24) which is a known angle from the right angled triangle ΔADU. Let the perpendicular at B to BC meet XA at Y. Then in the right angled triangle BYC we know $BY = 2AD = 2h_a$ and $BC = a$ is given. Now A lies on the perpendicular bisector of BY and also on the segment of the circle on YC subtending $180° - (B - C)$ at the circumference (note that $\angle YAC = 180° - \angle CAX = \angle B - \angle C$). This completes the construction of ΔABC.

Construction 36. Construct ΔABC given a, m_a, $B - C$.

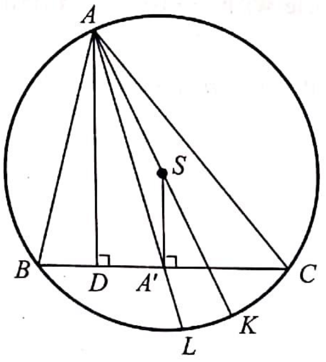


Fig. 4.100

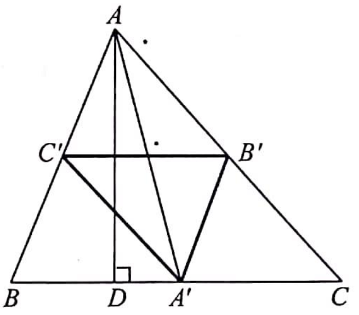


Fig. 4.101

See Fig. 4.100. Suppose ΔABC is the required triangle.

Let AS meet the circumcircle again at K (Fig. 4.100). We have $\angle A'SK = \angle DAK = \angle B - \angle C$ (Theorem 24). Therefore $\angle ASA' = 180° - \angle DAS = 180° - (B - C)$. If AA' meets the circumcircle again at L then $A'A \,.\, A'L = A'B \,.\, A'C$ (secant property). Therefore $A'L = (A'B \,.\, A'C)/(A'A) = (a^2/4m_a)$. $A'L$ is the third proportional to a and $4m_a$ which can be constructed. Now, construct the segment $AA' = m_a$ and on it construct the segment of a circle at which AA' subtends $180° - (B - C)$. S must lie on this arc. Produce AA' to L such that $A'L - (a^2/4m_a)$. S also lies on the perpendicular bisector of AL. Now the circle with centre S and radius SA cuts the perpendicular to SA' at the vertices B and C. ABC is the required triangle.

Construction 37. Given h_a, m_a, and A construct ΔABC.

As h_a and m_a are known, $\Delta ADA'$ may be constructed (Fig. 4.101). The median AA' subtends $\angle AB'A' = 180° - \angle A$ at B' as $A'B' \parallel AB$. Therefore B' lies on the circular arc on AA' at which AA' subtends $180° - \angle A$. Further the line joining the midpoints of AD and AA' should pass through B'. These informations determine B', from which ΔABC is easily constructed. ❑

Construction 38. Construct triangle ΔABC given A, m_b, m_c.

See Fig. 4.102. Now $\angle BAB' = \angle A$ and hence A lies on the segment of a circle on BB' at which BB' subtends $\angle A$. Produce BB' to X such that $B'X = BB'$. Then $B'CX = \angle A$ since

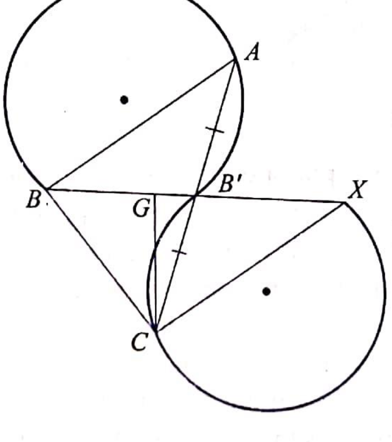


Fig 4.102

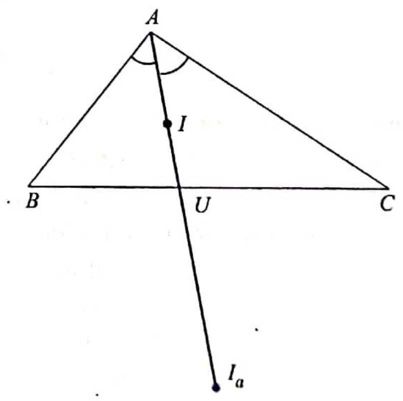


Fig. 4.103

$BCXA$ is a parallelogram (see construction 34). Therefore C lies on the segment of the circle on $B'X$ at which $B'X$ subtends $\angle A$. We find G on BB' from $BG : GB' = 2 : 1$. Having determined G, we note that C also lies on the circle with centre G and radius $(2/3)m_c$. Thus we can determine C. Now ΔABC is readily constructed. ❑

Construction 39. Given a, t_a, $b + c$, construct ΔABC.

Let $b + c = k$. (See Fig. 4.103). Divide AU internally and externally in the ratio $k : a = (b + c) : a$. Then we get the points I and I_a. The circle on II_a as diameter passes through B and C. Therefore BC is a chord through U of the circle on II_a as diameter, having the given length a, (Why?). This gives the required triangle ABC.

Note. The above problem has two solutions. ❑

Construction 40. Construct a triangle ABC, given the three points of intersection of the internal bisectors produced with the circumcircle.

We are given in position the three points P, Q and R (Fig. 4.104). The circumcircle of ΔPQR is the same as the circumcircle of ΔABC. As we have already seen P, Q, R are the midpoints of II_a, II_b and II_c respectively. We observe that AI is the common chord of the two circles (Q, QA) and (R, RI). So, AIP is perpendicular to QR and hence A is the second point of intersection of the altitude through P of the known triangle PQR. This determines A and similarly we find the other two vertices B and C. ❑

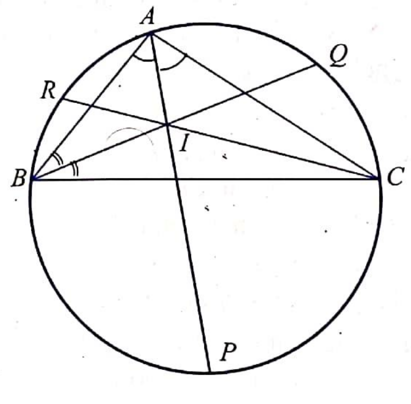


Fig. 4.104

Construction 41. Construct ΔABC given a, R, r.

Draw a circle of radius R and construct a chord BC of this circle having length a. Then $\angle A = \angle BPC$ for any point P on the corresponding segment of the circle. We know that $\angle BIC = 180° - (B/2 + C/2) = 90° + A/2$. This says that I should lie on the arc on BC at which BC subtends $90° + A/2$. Further I is at a distance r from BC. This determines I. Once $\angle BIC$ is known, ΔABC is easily constructed.

Note. The above problem has two solutions or one solution. ❑

Construction 42. Given A, $c + a$, r construct ΔABC.

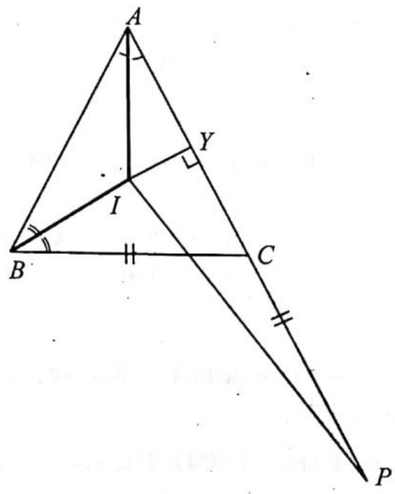


Fig. 4.105

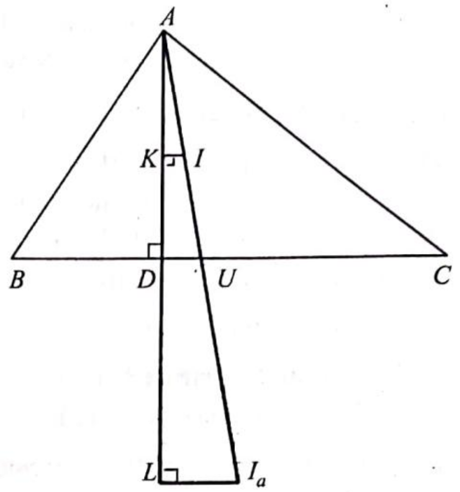


Fig. 4.106

(See Fig. 4.105). Produce AC to P such that $CP = BC$. In ΔAIP we know $AP = c + a$, $\angle IAP = A/2$ and the altitude IY through I. So, ΔAIP may be constructed. $\angle PBC = C/2$ and so $\angle PBI = C/2 + B/2 = 90° - A/2$. Therefore B lies on the arc of a circle on PI at which PI subtends $90° - A/2$. Also B lies on the tangent from A other than AP to the incircle (I, r). This fixes B. Now, C is the intersection of AP with the second tangent from B other than BA to the incircle (I, r). This completes the construction of ΔABC. ❑

Construction 43. Given a, h_a and $b + c$ construct ΔABC.

See Fig. 4.106. We have $ah_a = 2$ area of $\Delta ABC = (a + b + c)r$. This says that r is the fourth proportional to $a + b + c$, a, h_a, and hence can be constructed. I, I_a divide AU harmonically. Therefore K, L divide AD harmonically. Therefore given r, h_a the exradius r_a can be constructed. Now with a and $r_a - r$, using $2a = R - (r_a - r)$ we can construct R. Knowing c, h_a, R, the triangle is easily constructed. ❑

Construction 44. Given a triangle and a point on one of its sides, trisect the triangle by means of straight lines drawn through this point.

Let P be any given point on AC. We can construct t such that $3AP : AC = AB : t$. Let Q be on AB such that $AQ = t$. (See Fig. 4.107). Construct CT such that $3CP : CA = CB : CT$. Then we have $3AP . AQ - AB . AC$ and hence

$$\frac{1}{3} = \frac{AP . AQ}{AB . AC} = \frac{\Delta AQP}{\Delta ABC}$$

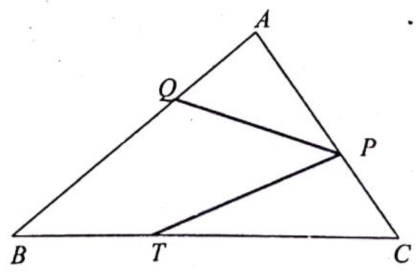


Fig. 4.107

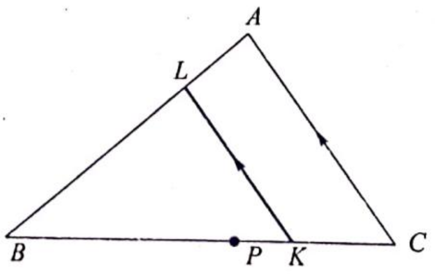


Fig. 4.108

Similarly $\dfrac{1}{3} = \dfrac{\Delta PTC}{\Delta ABC}$ and we see that PQ, PT are the required lines. ❑

Construction 45. Draw a triangle similar to a given triangle ABC and equal in area to p/q times the area of ΔABC.

Let P be a point on BC such that $BP/BC = p/q$ (If $p > 1$, p lies on BC produced). Construct the mean proportional BQ between BP and BC. Let K be the point on BC such that $BK = BQ$. Draw $KL \parallel CA$ meeting BA at L (Fig. 4.108).

Then
$$\frac{\Delta LBK}{\Delta ABC} = \frac{BK^2}{BC^2} = \frac{BP.BC}{BP^2} \quad \text{(by construction)}$$
$$= \frac{BP}{BC} = \frac{p}{q}.$$

Therefore ΔLBP is the required triangle. ❑

Construction 46. Construct a triangle similar to a given triangle ABC and equal in area to a second triangle DEF.

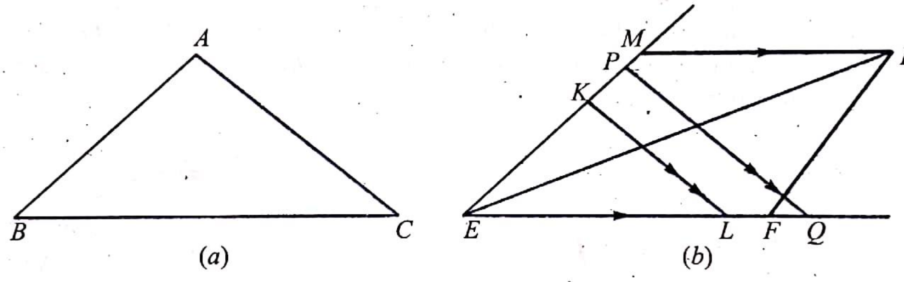


Fig. 4.109

On EF construct KEF similar to ΔABC. Draw $DM \parallel FE$ meeting EK (produced if necessary) at M (Fig. 4.109). Let EP be the mean proportional between EK and EM. Draw $PQ \parallel KF$ meeting EF at Q. Then

$$\frac{\Delta PEQ}{\Delta KEF} = \frac{EP^2}{EK^2} = \frac{EK.EM}{EK^2} = \frac{EM}{EK} = \frac{\Delta DEF}{\Delta KEF}$$

or ΔPEQ is the required triangle. ❑

Construction 47. Divide AB internally and externally at X, Y respectively such that $AX^2 = AB - XB$ and $AY^2 = AB \cdot YB$ (medial section).

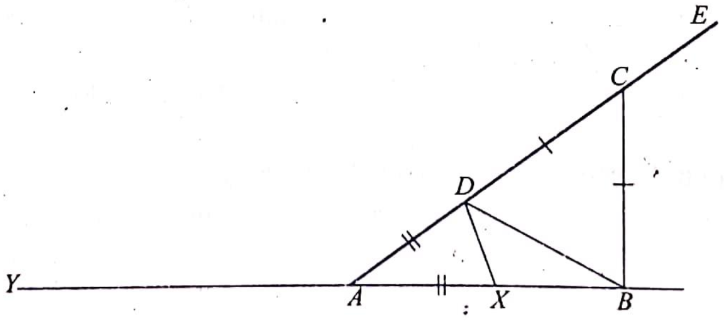


Fig. 4.110

Draw BC perpendicular to AB such that $BC = (1/2)\ AB$. Take D on CA such that $CD = CB$. Take X on AB such that $AX = AD$. We have $AB^2 + BC^2 = AC^2 = (AD + DC)^2$

$$= AD^2 + DC^2 + 2AD\ .\ DC$$

$$= AX^2 + BC^2 + 2AX\ .\ BC$$

Therefore $\quad AB^2 = AX^2 + 2AX\ .\ BC = AX^2 + AX\ .\ AB$

Therefore $\quad AX^2 = AB^2 - AXAB = AB(AB - AX) = AB\ .\ XB.$

Thus X is the required point. For external division, extend AC to E such that $CE = CB$. Take Y on BA produced (Fig. 4.110) such that $AY = AE$. Then one can easily check that Y is the required point. ❑

Construction 48. On a given base BC construct an isosceles triangle ABC such that $\angle B = \angle C = 2\angle A$.

Divide BC externally at Y in medial section, *i.e.*, $BY^2 = BC\ .\ YC$. (Fig. 4.111). Construct an isosceles triangle ABC on BC such that $AB = AC = BY$. We have $AC^2 = BY^2 = BC\ .\ YC$. Therefore CA is a tangent to the circle YBA at the point A. Hence $\angle CAB = \angle AYB$ (angle in the alternate segment) $= \angle BAY$ (by construction). Therefore $\angle ABC = \angle B = 2\angle CAB = 2\angle A$. ❑

Note. 1. $\angle B = \angle C = 72°$ and $\angle A = 36°$.

2. Given one of the equal sides AB, describe an isosceles ΔABC with $\angle B = \angle C = 2\angle A$. For this divide AB internally at X such that $BX^2 = BC\ .\ XC$. Then AX is the base for the required triangle ABC.

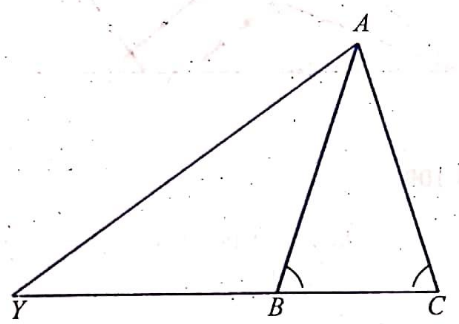


Fig. 4.111

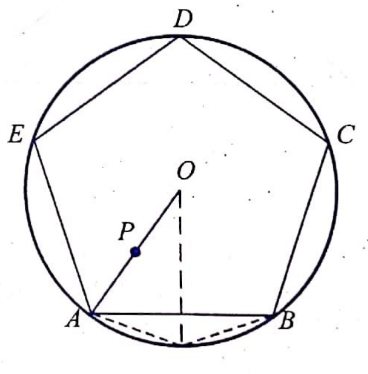


Fig. 4.112

Construction 49. Inscribe a regular pentagon in a given circle.

Let A be any point on the given circle with centre O. Find P on AO such that $AP^2 = AO\ .\ PO$. Let AQ be a chord equal in length to AP. Then by Note(2) of the previous construction, $\angle AOQ = 36°$. Take QB again equal to AP. This gives $\angle QOB = 36°$ and so $\angle AOB = 72°$. So, AB is a side of the required regular pentagon. We continue this construction to get C, D, and E (Fig. 4.112). ❑

Note. A, Q, B in Fig. 4.112 are three consecutive vertices of a regular decagon inscribed in the same circle.

Construction 50. Construct ΔABC given $2s$, A and t_a.

See Fig. 4.113. As $AZ_a = s$, from the given data we may construct ΔAI_aZ_a. Take U on AI_a such that $AU = t_a$. Draw the excircle opposite to A, namely $(I_a, I_a Z_a)$. Now, AZ_a

the other tangent from A to this excircle and a tangent from U to the same circle are the sides of ΔABC. ❑

Note. The above problem has two, one or no solutions.

Construction 51. Construct a triangle ABC given its altitudes h_a, h_b and h_c.

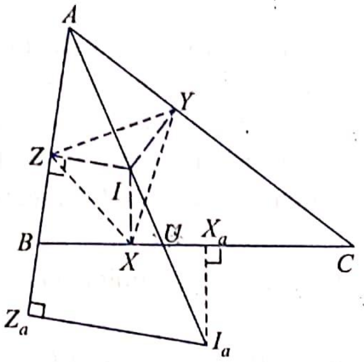


Fig. 4.113

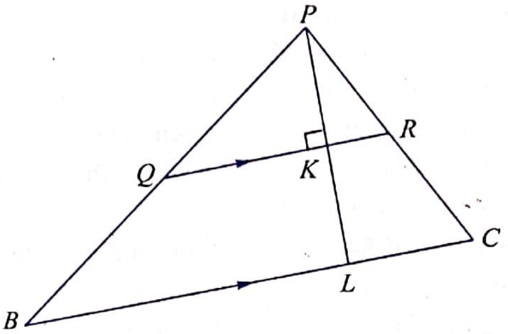


Fig. 4.114

We have $ah_a = bh_b = ch_c = 2\Delta ABC$.

$\therefore$ $\dfrac{a}{h_b} = \dfrac{b}{h_a} = \dfrac{c}{k}$ where $k = \dfrac{h_a h_b}{h_c}$

$\therefore$ $\dfrac{h_c}{h_a} = \dfrac{h_b}{k}$ and k can be constructed.

ΔABC ||| ΔPQR where $QR = h_b$, $RP = h_a$, $PQ = k$. (Fig. 4.114). Let PK be the altitude through P for ΔPQR. Take L on PK such that $PL = h_a$. Let the parallel through L to meet PQ, PR at B, C respectively. Then if A is taken at P, we see that ABC is the required triangle. ❑

Note. For ΔPQR to exist we must have

$$h_a + h_b > k > h_a - h_b. \quad \text{In other words}$$

$$h_a + h_b > \frac{h_a h_b}{h_c} > h_a - h_b \quad \text{or equivalently}$$

$$\frac{1}{h_b} + \frac{1}{h_a} > \frac{1}{h_c} > \frac{1}{h_b} - \frac{1}{h_a}.$$

Construction 52. Construct a triangle given the points where the altitudes produced meet the circumcircle.

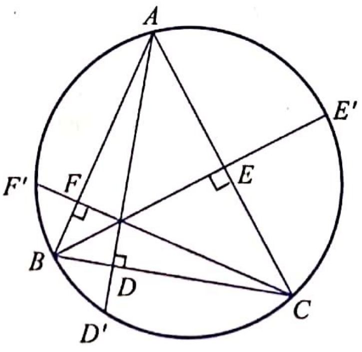


Fig. 4.115

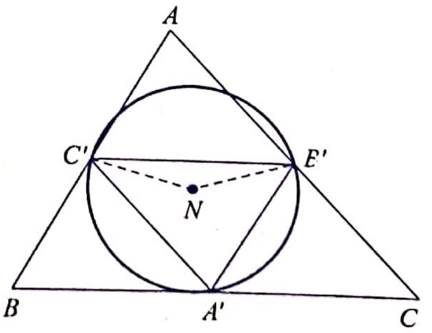


Fig. 4.116

The given points D', E', F' determine the circumcircle of ΔABC. Further, A, B and C are the midpoints of the arcs $E'F'$, $F'D$ and $D'E'$. (Fig. 4.115).

Construction 53. Given in position the nine-point centre N and one vertex A and the directions of the internal bisector and of the altitude passing through the given vertex A, construct ΔABC.

We know that the circumcentre S is on AS which is symmetric to the altitude through A with respect to the given bisector through A. Also N is the midpoint of SH implies that S is on the symmetric of the altitude with respect to the nine-point centre. These informations determine the circumcentre S. This in turn enables us to locate H. If AH meets the circumcircle (S, SA) at D' then the perpendicular bisector of HD' meets the circumcircle at B and C.

Construction 54. Construct a triangle given the position of the nine-point centre and one angle both in magnitude and position.

Suppose we are given the angle at A. We note that $\angle B'NC' = 2\angle B'A'C' = 2A$. (Fig. 4.116). Therefore the isosceles $\Delta NB'C'$ is a triangle with a known vertex and known three angles. We use the following lemma.

Lemma. If the vertex A of a variable triangle ABC is kept fixed and B moves on given straight line, such that ΔABC always remains similar to a given triangle then C describes a straight line.

Let ABC be the position of the variable triangle when BC is on the given straight line. Let AB_1C_1 be any other position. (Fig. 4.117). Then $\angle ACB_1 = \angle AC_1B_1$ and hence the quadrilateral AB_1CC_1 is cyclic. Therefore $\angle ACC_1 = \angle AB_1C_1$ and hence $\angle BCC_1 = \angle BCA + \angle ACC_1 = \angle BCA + \angle AB_1C_1$ = Fixed Angle. Therefore CC_1 makes a constant angle with the given line and hence C moves on a straight line. This proves the lemma.

Now, by the lemma, when N remains fixed, B' moves on the fixed line AC, such that $\Delta NB'C'$ have constant angles $\angle N$, $\angle B$, $\angle C$, we see that C' moves on another straight line; this locus of C' determines the position of C' on AB. Draw the quadrilateral $AC'NB'$ such that $AC'NB' = 2A$. From the quad. $AC'NB'$ passage to ΔABC is immediate. ❑

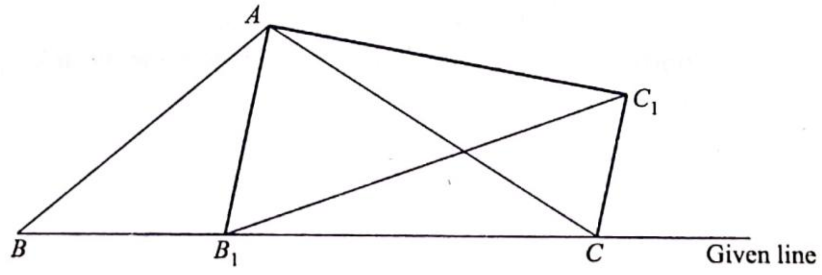


Fig. 4.117

Construction 55. Construct a quadrilateral $ABCD$ given the four sides AB, BC, CD, DA and the line joining the midpoints of AB and CD.

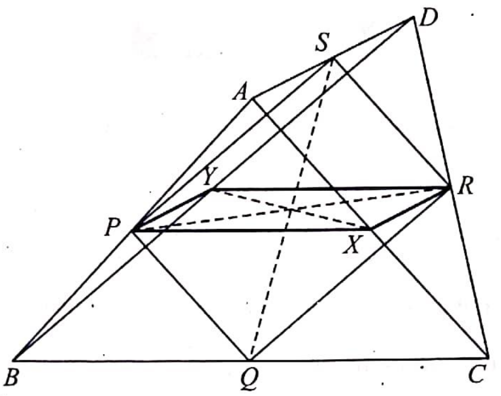


Fig. 4.118

We can construct the parallelogram $PXRY$ since we know the diagonal PR, side $PX = (1/2)\ BC$, side $XR = (1/2)\ AD$. See Fig. 4.118. This gives the diagonal XY of the parallelogram $QXSY$, in which we know the side $QX = (1/2)\ AB$, $XS = (1/2)\ CD$. Therefore the parallelogram $QXSY$ may be constructed. Now, the four points P, Q, R, S are located and the quadrilateral $ABCD$ is easily constructed. ❑

Construction 56. Construct a quadrilateral given two opposite angles, the diagonals and the angle between the diagonals.

Suppose we are given $\angle B, \angle D, AC, BD$ and the angle $\angle BOC$ between the diagonals. (Fig. 4.119). Construct the two circular segments on AC on which B and D should lie. Now, just find two points B, D on these two circular arcs constructed such that BD has the given length and makes the given angle with AC. (see construction 16). Our problem is solved now. ❑

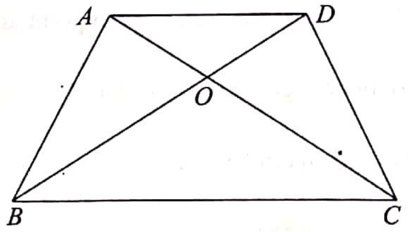


Fig. 4.119

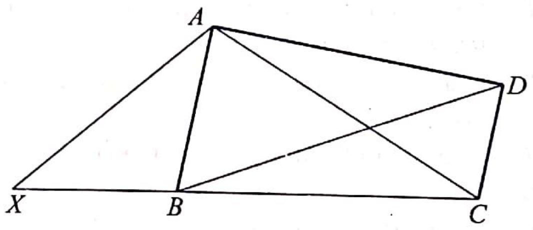


Fig. 4.120

Construction 57. Construct a cyclic quadrilateral $ABCD$ given its sides AB, BC, CD and DA.

Draw the segment BC and take X on CB produced such that $BX = (AB\ .\ CD)/AD$. (Fig. 4.120). Now, $ABCD$ is cyclic implies that

$$\angle ABX = \angle ADC, \quad \frac{AB}{BX} = \frac{AB}{(AB\ .\ CD)/AD} = \frac{AD}{CD}$$

Therefore $\Delta AXB \,|||\, \Delta ACD$ and hence $\dfrac{AX}{AC} = \dfrac{AB}{AD}$

Therefore A lies on the circle on KL as diameter where K, L divide XC internally and externally, in the ratio AB/AD. Also A lies on the circle with centre B and radius BA. This determines ΔABC from which we easily pass on to the quadilateral $ABCD$. ❑

EXERCISE 4.5

1. Draw a circle with centre O. Choose a point A on the circle; cut off chord AB, BC, CD, DE, EF each equal to the radius. Prove that $AF = AB$ and that $ABCDEF$ is a regular hexagon.
2. On a given line segment. BC, construct an equilateral triangle ABC. Bisect the angles B and C by straight lines meeting at S. Draw SD, SE parallel to AB, AC respectively to meet BC at D, E. Prove that D, E trisect BC, *i.e.*, $BD = DE = EC$.
3. Given a line segment x units long, construct one of length x^2 units.
4. Divide a straight line AB in the ratio $\sqrt{2} : \sqrt{3}$.
5. Divide a line segment internally at X and externally at Y, so that $AX^2 : XB^2 = AY^2 : YB^2 = 2 : 5$.
6. S is a given circle with centre O. Through a given point A, draw a straight line to cut S at X, Y such that $XY = BC =$ a given line segment (in length).
7. Draw a circle touching a given circle and a given straight line at a given point P.
8. Draw a circle to touch a given line AB and a given circle at a given point P.
9. A and B are given points; draw a circle S with centre A so that the tangent to S from B is of given length (less than AB).
10. In a given circle, place a chord of given length. How many such chords can be drawn?
11. Inscribe a circle in a given triangle.
12. Draw a circle through two given points A, B to touch a given straight line CD.
13. Draw a circle to touch two given straight lines OA, OB and pass through a given point C.
14. Draw a circle to touch a given circle (centre C and radius r) and also to touch two given straight lines OA, OB.
15. Draw a circle through two given points A, B to touch a given circle.
16. Draw a circle, with its centre on a given straight line, to pass through a given point and touch a given circle.
17. Draw a circle to pass through a given point A, to touch a given straight line BC and a given circle.
18. Draw a circle through a given point A to touch two given circles C_1, C_2.
19. Draw a circle to touch three given circles.
20. Let A, B be two given points on a circle S, l is a given straight line and C is a given point on it; find a point M on the circle such that if AM, BM meet l at P, Q then $CP/CQ = \lambda$, a given ratio.
21. Construct a triangle so that its sides pass through three noncollinear points and be divided by these points internally in given ratios.
22. Draw a circle tangent to two concentric circles and passing through a given point.
23. Inscribe a square in a given quadrilateral.
24. In a given triangle inscribe a parallelogram having a given angle and having its adjacent sides in a given ratio.
25. Construct ΔABC, given R, a, $(b + c)b$.
26. Construct ΔABC, given in position (*i*) I_a, I_b, I_c (*ii*) I, I_b, I_c (*iii*) S, I, I_a.
27. Construct a quadrilateral given the four sides and the sum of two opposite angles.

4.6 SOME GEOMETRIC GEMS

In this section we try to give an assorted collection of beautiful problems in the geometry of straight lines, triangles and circles in a plane.

Problem 1. Let ABC be an acute angled triangle and D any point on BC. Find points E, F on the sides CA, AB of ΔABC such that the perimeter of ΔDEF is of minimum length.

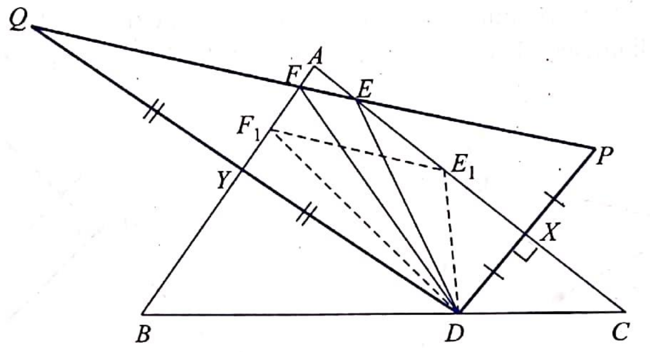


Fig. 4.121

Draw DX, DY perpendicular to CA, AB (See Fig. 4.121) and produce them to P and Q such that $DX = XP$ and $DY = YQ$. Let PQ cut the sides CA, AB at E, F-respectively. We claim that DEF is the required triangle. For, by construction $DE = EP$ and $DF = QF$. Hence the perimeter of $\Delta DEF = DE + EF + FD = PE + EF + FQ = PQ$. If E_1 and F_1 are any two points on CA, AB respectively then $DE_1 = E_1P$ and $DF_1 = F_1Q$ and hence perimeter of $\Delta DE_1F_1 = PE_1 + E_1F_1 + F_1Q \geq PQ$ (note that the distance between any two points in a plane is the minimum along the straight line joining the two points!). Hence E, F are the required points. ❑

Problem 2. Given a circle (S, R) with centre S and radius R show that an infinite number of triangles may be inscribed in it, having their centroid at a given point within the circle.

Let A be any point on the given circle and G be the given point inside the circle. Produce AG to A' such that $AG : GA' = 2 : 1$. See Fig. 4.122. Let the perpendicular to SA' cut the circle at B and C. Then ABC is a triangle inscribed in the given circle having G as its centroid. Now, if A' lies outside the circle, then we do not have any solution. We note that as A moves, the locus of A' is a circle. In fact the locus of A' is the circle $(N, R/2)$ where N is the point on SG dividing it externally in the ratio $3 : 1$. (Recall that

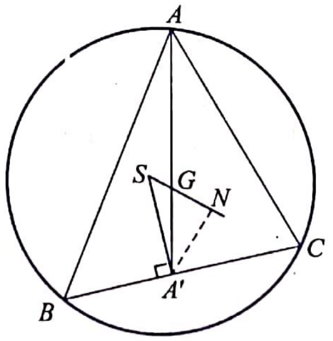


Fig. 4.122

the nine-point centre N of ΔABC lis on SG and satisfies $SG : GN = 2 : 1$). If this circle $(N, R/2)$ lies within the circle, A may be chosen as any point on (S, R) and we get a solution. Otherwise, we have an arc of the circle (S, R) on which A should not be chosen, in order that we get a solution. ❑

Problem 3. ABC is a right angled triangle, right angled at B. The triangle rotates about B such that C and A always lie on two perpendicular lines OX, OY respectively. Find the locus of the centroid of ΔABC.

See Fig. 4.123. Now, by our hypothesis we note that the quadrilateral $OCBA$ is cyclic for which AC is a diameter. Therefore the midpoint B' of CA lies on the perpendicular bisector

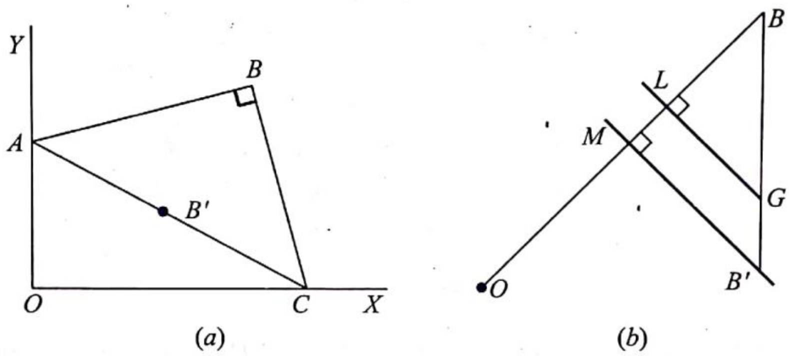


Fig. 4.123

of the fixed line segment OB. The centroid G of ΔABC is on BB' dividing it in the ratio $2 : 1$. Now the locus of B' is the perpendicular bisector of OB implies that the locus of G is a straight line GL parallel to the perpendicular bisector of OB [Fig. 4.123(b)]; such that

$$\frac{BL}{BM} = \frac{2}{1}.$$ ❑

Problem 4. If $ABCD$ is a rhombus and P is equidistant from B and D then A, C, P are collinear and further $PC \,.\, PA = PB^2 - AB^2$.

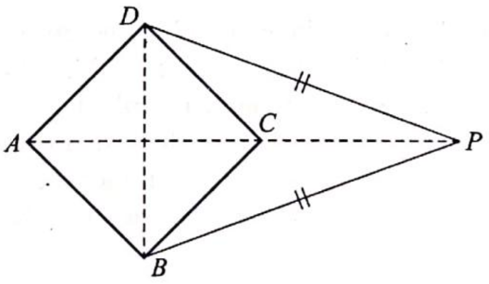


Fig. 4.124

$PB = PD$ implies that P is on the perpendicular bisector of BD. Since $ABCD$ is a rhombus, AC is the perpendicular bisector of BD and hence A, C and P are collinear. Also $PA \,.\, PC = (PO + OA)\,(PO - OC) = (PO + OA)\,(PO - OA)$

$$= OP^2 - OA^2$$
$$= OP^2 + OB^2 - (OC^2 + OB^2)$$
$$= PB^2 - BC^2 = PB^2 - AB^2$$ ❑

Problem 5. If ABC is an equilateral triangle and P lies on the arc BC of the circumcircle of ΔABC then $PA = PB + PC$.

Quadrilateral $ABPC$ is cyclic and Ptolemy's theorem applied to this quadrilateral gives $BC \,.\, PA = AB \cdot PC + AC \,.\, BP$. Here we have $AB = BC = CA$.

Therefore, $PA = PB + PC$. (Fig. 4.125). ❑

Problem 6. (Erdös-Mordell Theorem). If O is any point inside a triangle ABC and P, Q, R are the feet of the perpendiculars from O upon the respective sides BC, CA, AB of ΔABC then $OA + OB + OC \geq 2(OP + OQ + OR)$.

Let A_1, A_2 be the feet of the perpendiculars from R and Q on BC; similarly B_1, B_2 are the feet of the perpendiculars from P and R on CA; C_1, C_2 are the feet of the perpendiculars from Q and P on AB (Fig. 4.126).

In the triangles PRA_1 and OBR we have

$$\angle PA_1R = \angle ORB = 90°$$

$$\angle PRA_1 = \angle RPO \quad \text{(alt angles)}$$

$$= \angle OBR \text{ (angles in the same segment of the circle } OPBR\text{)}$$

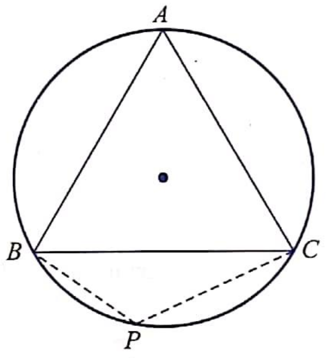


Fig. 4.125

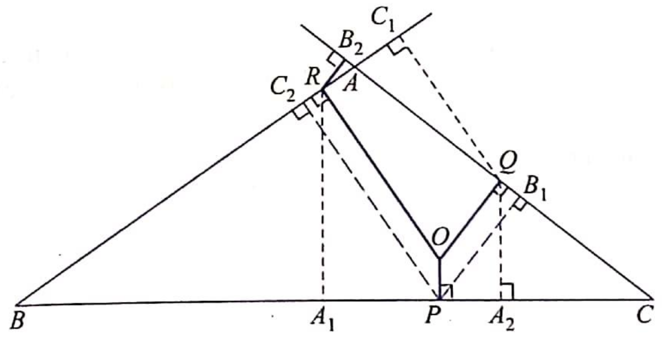


Fig. 4.126

$$\therefore \quad \Delta PRA_1 \,|||\, \Delta OBR \quad \text{and hence} \quad \frac{PA_1}{PR} = \frac{OR}{OB} \qquad (1)$$

It is clear from Fig. 4.126, that $A_1A_2 \leq RQ$, $B_1B_2 \leq PR$ and $C_1C_2 \leq PQ$. Similarly, we have

$$\frac{PA_2}{PQ} = \frac{OQ}{OC}, \quad \frac{B_1Q}{PQ} = \frac{OP}{OC}, \quad \frac{QB_2}{QR} = \frac{OR}{OA}, \quad \frac{GR}{QR} = \frac{OQ}{OA}$$

$$\frac{RC_2}{RP} = \frac{OP}{OB} \qquad (2)$$

Using (2) in $OA + OB + OC \geq OA\left(\frac{A_1A_2}{RQ}\right) + OB\left(\frac{B_1B_2}{PR}\right) + OC\left(\frac{C_1C_2}{PQ}\right)$

we get, $OA + OB + OC \geq OA\left(\frac{A_1P + PA_2}{RQ}\right) + OB\left(\frac{B_1Q + QB_2}{PR}\right)$

$$+ OC\left(\frac{C_1R + RC_2}{PQ}\right) + \frac{OC}{PQ}\left(\frac{QR \cdot OQ}{OA} + \frac{RP \cdot OP}{OB}\right)$$

$$= \frac{OA}{RQ}\left(\frac{PR \cdot OR}{OB} + \frac{PQ \cdot OQ}{OC}\right) + \frac{OB}{PR}\left(\frac{PQ \cdot OP}{OC} + \frac{QR \cdot OR}{OA}\right)$$

$$= OP\left(\frac{OB}{PR} \cdot \frac{PQ}{OC} + \frac{OC \cdot PR}{PQ \cdot OB}\right) + \text{similar terms}$$

$$\geq 2(OP + OQ + OR) \quad \{\text{since if } x > 0,\ x + 1/x \geq 2\}$$ ❑

Problem 7. Given an acute angled triangle ABC, find points D, E, F, on the sides BC, CA, AB of ΔABC such that the perimeter of ΔDEF is a minimum.

For a given point D on BC, this problem is solved in problem 1 of this section. By construction (see Fig. 4.121) we have $AP = AQ = AD$. Also, $\angle DAX = \angle XAP$ and $\angle DAY = \angle YAQ$ and $\angle DAX + \angle DAY = \angle A$.

Therefore, $\angle QAP = 2\angle A$. This means that for any choice of die point D on BC, $\angle QAP = 2\angle A$ = constant in the isosceles triangle AQP. Further QP = permeter of ΔDEF and therefore the perimeter of ΔDEF is a minimum when the side QP of the isosceles triangle AQP with constant vertical angle is a minimum. This happens when the equal sides have minimum length. But $AQ = AP = AD$ and AD is a minimum when D is the foot of the perpendicular from A on BC. Thus the required triangle is the (pedal) orthic triangle DEF. ❑

Problem 8. Let F be any point on the side AB of ΔABC. D be the intersection of BC with the straight line $AD \parallel FC$ through A. Similarly, let E be the intersection of CA with the line $BE \parallel FC$ through B. Prove that $\frac{1}{AD} + \frac{1}{BE} = \frac{1}{CF}$.

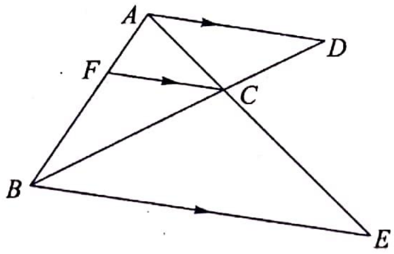


Fig. 4.127

$\Delta CAF \parallel\!\parallel \Delta EAB$ since $CF \parallel EB$ and $\Delta CBF \parallel\!\parallel \Delta DBA$. Therefore, we get

$$\frac{CF}{EB} = \frac{AF}{AB} \quad \text{and} \quad \frac{CF}{DA} = \frac{BF}{BA}$$

Adding we get $CF\left(\frac{1}{BE} + \frac{1}{AD}\right) = \frac{AF + FB}{AB} = 1$

$$\therefore \qquad \frac{1}{AD} + \frac{1}{BE} = \frac{1}{CF}$$ ❑

Problem 9. If a polygon is inscribed in a circle and a second polygon is cricumscribed by drawing tangents to the circle at the vertices of the first polygon, then the product of the perpendiculars on the sides of the first, from any point on the circle, equals the product of the perpendiculars from the same point to the sides of the second.

This problem will illustrate how a degenerate special case, on repeated applications, may prove the general case. We consider the special case now. See Fig. 4.128. Suppose AB is a chord of a circle and the tangents at A and B meet at C. P is any point on the circle and PL, PM, PN are the perpendicular from P on the sides AB, BC, CA of ΔABC. Then we claim that $PM \cdot PN = PL^2$.

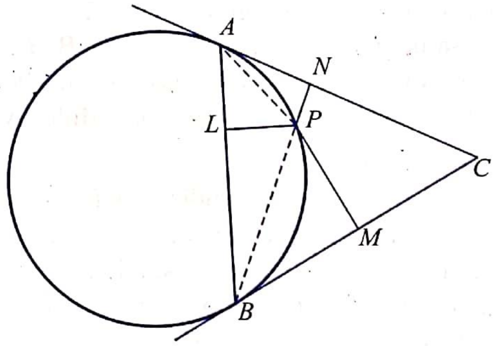


Fig. 4.128

Proof of claim: We have, $\Delta PAN \,|||\, \Delta PBL$

$$\therefore \qquad \frac{PN}{PL} = \frac{PA}{PB}$$

Again, $\Delta PBM \,|||\, \Delta PAL$ and

$$\frac{PA}{PB} = \frac{PL}{PM}.$$ Thus, we have

$$\frac{PA}{PB} = \frac{PL}{PM} = \frac{PN}{PL} \quad \text{or} \quad PM \cdot PN = PL^2$$

This ancillary result that we have just now proved is the problem 9, for a two sided polygon!

Now, consider an n sided polygon $A_1A_2A_3 \ldots A_n$, inscribed in a circle and let B_1, B_2 ... B_n be the polygon circumscribing the same circle got by drawing the tangents at the vertices $A_1, A_2, \ldots A_n$ (Fig. 4.129).

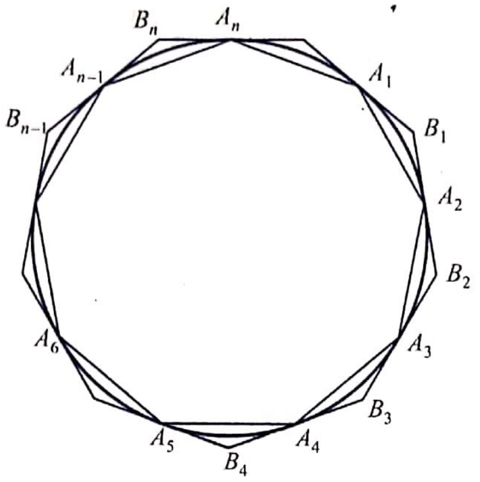


Fig. 4.129

Let the tangents at A_1 and A_2 meet at B_1; those at A_2 and A_3 meet at B_2 and so on. P be any point on the circle with $p_1, p_2, \ldots, p_n$ being the perpendiculars from P on the sides $A_1 A_2$, $A_2 A_3 \ldots A_{n-1} A_n$, $A_n A_1$ of the polygon $A_1A_2 \ldots A_n$. Also let $q_1, q_2, \ldots q_n$ be the perpendiculars from the same point P on the sides B_1B_2, $B_2B_3, \ldots B_{n-1} B_n$ of the outer polygon $B_1 B_2 \ldots B_n$. Then by the special case discussed in the beginning of the solution, we see that $p_1^2 = q_n q_1$, $p_2^2 = q_1q_2 \ldots$ and $p_n^2 = q_{n-1} q_n$. Multiplying we see that

$$p_1p_2 \cdots p_n = q_1 q_2 \cdots q_n.$$

Problem 10. The algebraic sum of the perpendiculars from any point to the sides of a regular polygon of n sides is a constant and is equal to n times the *apothem*. (*i.e.*, the line drawn from the centre of the polygon perpendicular to a side). We may attach signs to the perpendiculars such that for points within the polygon, the perpendiculars are all positive. Let 'a' denote the length of a side of the regular polygon and $h = OA$ (see Fig. 4.130) be the apothem. Then the polygon is made up of the n triangles OA_1A_2, $OA_2A_3, \ldots, OA_nA_1$. Each of these triangles has area equal to (1/2) ah. Hence the area of the polygon is given by $\Delta = (1/2)\, nha$. If P is any point on the plane of the polygon, we see that the area of the polygon $\Delta = (1/2)\, a \sum_{i=1}^{n} h_i$ where h_i is the algebraic perpendicular distance of P from the side A_iA_{i+1}. Thus we get $nh = h_1 + h_2 + \ldots + h_n$ or the algebraic sum of the perpendiculars from any point P to the sides of a regular polygon is equal to n times the apothem. ❑

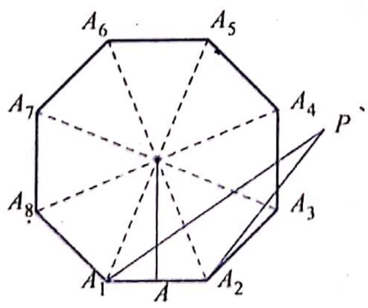


Fig. 4.130

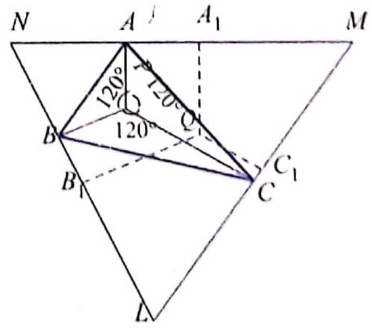


Fig. 4.131

Remark. Applying problem 10 to an equilateral triangle we see that the sum of perpendiculars from a point to the sides equals the altitude; also for a square, the sum of the perpendicular distances equals twice the side of the square.

Problem 11. If no angle of a triangle ABC is greater than or equal to 120°, then the point P inside the ΔABC such that $PA + PB + PC$ is a minimum is the Fermat point of ΔABC making 120° with each side of ΔABC.

The Fermat point of a triangle ABC in which no angle is bigger than 120° is the point P inside the triangle such that $\angle BPC = \angle CPA = \angle APB = 120°$. (Fig. 4.131). If we draw equilateral triangles outwardly on the sides AB and AC, say ΔAXB and ΔAYC, then P is the other point of intersection of the circles AXB and AYC. (Fig. 4.132) Now, having located the Fermat point P inside ΔABC, draw MN, NL and LM perpendicular to PA, PB and PC respectively. By construction the quadrilaterals $PANB$, $PBLC$ and $PCMA$ are all cyclic. Hence, each angle of ΔLMN is 60° or ΔLMN is an equilateral triangle. By the previous problem (Problem 10, Remark) we have $PA + PB + PC = h =$ the altitude of ΔLMN. If Q is any other point inside ΔABC, let A_1, B_1, C_1 be the feet of the perpendiculars from Q on the sides MN, NL and LM respectively.

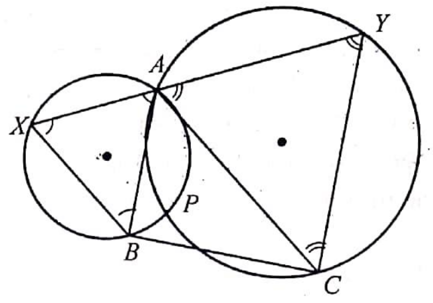


Fig. 4.132

Again by problem 10 (remark) we have $QA_1 + QA_2 + QA_3 = h =$ altitude of ΔLMN. But it is clear from the figure (Fig. 4.131) that $QA + QB + QC > QA_1 + QB_1 + QC_1 = h = PA + PB + PC$. This means that the Fermat point P is our required point. We note that at best only one of the pairs of points (A, A_1), (B, B_1), (C, C_1) can be coincident pair of points and hence $QA + QB + QC > QA + QB_1 + QC_1$. ❑

Problem 12. If ABC is a triangle in which no angle is bigger than equal to 120° and equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are constructed outwardly on the sides AB, BC, CA of ΔABC then the lines AA', BB', CC' concur at the *Fermat point P* of ΔABC, and further $AA' = BB' = CC'$. (Fig. 4.133).

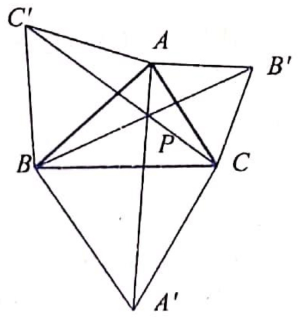


Fig. 4.133

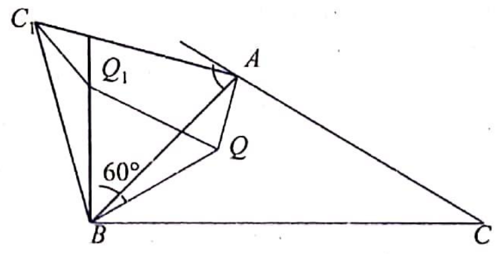


Fig. 4.134

Suppose Q is any point within ΔABC. Now, rotate ΔQAB about B through 60° to get ΔC_1Q_1B (Fig. 4.134).

We note that ΔAC_1B is isosceles as $AB = C_1B$ and further $\angle ABC_1$ = angle of rotation = 60°. Therefore, ΔAC_1B is equilateral. This means that irrespective of the position of Q, C_1 must be the third vertex of the equilateral triangle on AB, drawn outwardly, and hence C_1 coincides with C'. Again, $\angle QBQ_1 = 60°$ and $QB = Q_1B$ implies that ΔBQQ_1 is also equilateral.

This means that $QA + QB + QC = C_1Q_1 + Q_1Q + QC$

= the length of the polygonal path $CQ\,Q_1\,C_1$. (1)

This polygonal path joining C and C_1 has minimum length when Q and Q_1 lie on the line segment CC_1, in which case Q becomes the Fermat point of ΔABC (in view of (1)). When CQQ_1C_1 is a straight line, we should have $\angle BQQ_1 = 60°$ and hence $\angle BQC_1 = 60°$; this implies that the circumcircle of $\angle AC_1B$ passes through Q and this fixes the position of Q, on CC'. Therefore, the Fermat point P lies on CC' and $CC' = PA + PB + PC$. By symmetry, the Fermat point P also lies on AA' and BB'. Further $AA' = BB' = CC = PA + PB + PC$. ❑

Note. There is another beautiful solution to Fermat's problem using Ptolemy's theorem.

Let ABC be any triangle. Let B and C be acute angles of ΔABC. Construct the equilateral triangle $BA'C$ on BC and consider its circumcircle. (Fig. 4.135). If P is on the minor arc BA' of the circle $BA'C$ then by Ptolemy's theorem,

$PB \,.\, CA' + BC \,.\, PA' = PC \,.\, BA'$ which gives $PB + PA' = PC$ and hence $PB + PC > PA'$. Similarly if P is on the minor arc CA', we get $PB + PC > PA'$. If P is on the minor arc BC, then we have

$$PB \,.\, CA' + PC \,.\, BA' = BC \,.\, PA' \text{ or}$$

$$PB + PC = PA'$$

∴ Unless P lies on the minor arc BC we have

$$PB + PC > PA' \text{ or } PA + PB + PC > PA + PA'$$

Now $PA + PA'$ is a minimum only when P lies on AA' and in which case $PA + PA' = AA'$. Thus the minimum of $PA + PB + PC$ is AA' and it occurs when P is at the intersection of AA' and the circumcircle of $\Delta BA'C$. In case $\angle BAC = 120°$, A coincides with P. If $\angle BAC > 120°$. A lies within the circle $BA'C$ and A is still the Fermat point of ΔABC.

❑

Problem 13. If $C'AB$, $A'BC$ and $B'CA$ are the equilateral triangles drawn outwardly on the sides of a given triangle ABC then the centres X, Y, Z of the equilateral triangles form another equilateral triangle.

Let us draw the major arcs of the circles $C'AB$, $A'BC$ and $B'CA$. Let PQ be any line segment through A intercepted by the circles $C'AB$ and $B'CA$ at P, Q respectively (Fig. 4.136). Let PB and QC meet at R. We note that $\angle QPB = \angle PQC = 60°$ and hence $\angle BRC = 60°$. Therefore, R lies on the circle $A'BC$. Further ΔPQR is equilateral. From the centres X and Y, drop the perpendiculars XL, YK on the side QR of ΔPQR. Then L and K are the midpoints of the chords RC and CQ. Draw $XM \perp YK$ so that $XMKL$ is a rectangle. Now, K and L being the midpoints of QC and CR we see that $QR = 2KL$. Therefore QR is the largest when $LK = XM = XY$ (Fig. 4.136); *i.e.*, when M coincides with Y or $QR \parallel XY$. Further maximum $QR = 2XY$. For a similar reason PQ is a maximum

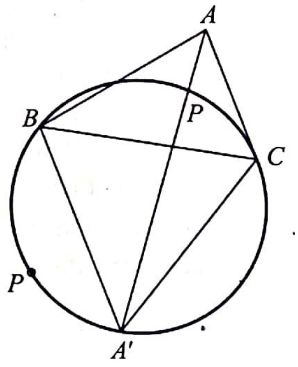


Fig. 4.135

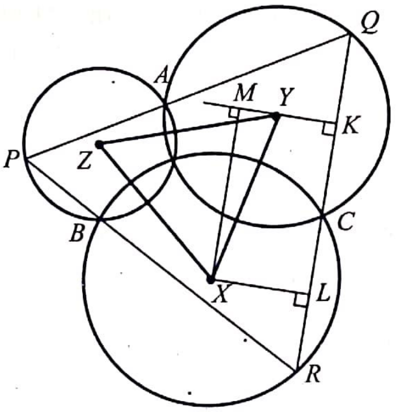


Fig. 4.136

when $PQ \parallel YZ$ and $PQ = 2YZ$ and maximum PR is twice ZX. But for any choice of P on the major arc of circle $C'AB$, we always have PQR as an equilateral triangle. In particular when ΔPQR is the largest, we have $PQ = 2YZ = QR = 2XY = RP = ZX$. Hence ΔXYZ is equilateral. ❑

Problem 14. Consider the two geometric transformations Rot (O_1, a) and Rot (O_2, β) where Rot (A, θ) means rotation about the point A through an angle θ. Find the sum of the two rotations Rot (O_1, α) and Rot (O_2, β).

Consider a line segment AB. Under Rot (O_1, α), AB is transformed into A_1B_1. We have $AB = A_1B_1$ in length and the angle between A_1B_1 and AB is α the angle of rotation. Under Rot (O_2, β) A_1B_1 is transformed into A_2B_2, with $A_2B_2 = A_1B_1$ in length and the angle between A_2B_2 and A_1B_1 is β (Fig. 4.137). Therefore, the angle between A_2B_2 and AB is $\alpha + \beta$ and hence the transformation Rot (O_2, β). Rot (O_1, α) preserves the length of line segments and rotates them through ($\alpha + \beta$). This means that Rot (O_2, β) Rot (O_1, α) is again a rotation through ($\alpha + \beta$) about some point O, unless $\alpha + \beta = 360°$ in which case it becomes a translation. We may get the centre of rotation of Rot (O_2, β) Rot (O_1, α), when $\alpha + \beta \neq 360°$, as follows.

Under the sum of the two rotations, O_1 goes to a point O_1' such that $O_2O_1, = O_2O_1'$ and $\angle O_1O_2O_1' = \beta$ (Fig. 4.138). If O_2'' is the point on the ray through O_1 making an angle α

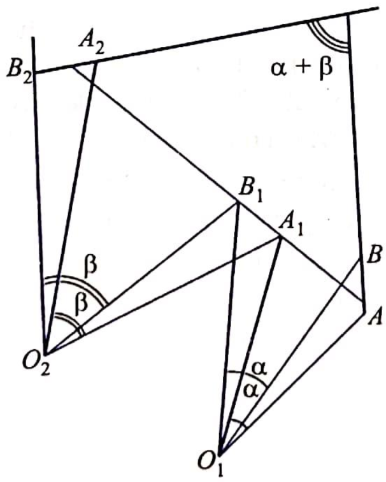


Fig. 4.137

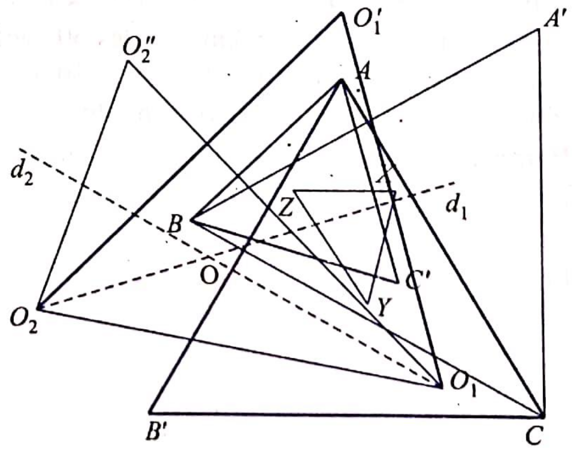


Fig. 4.138

1with O_2O_1 as in Fig. 4.138, such that $O_1O_2'' = O_2O_1$ then O_2 is the image of O_2'' under Rot (O_2, β) Rot (O_1, α). In fact Rot (O_1, α) takes O_2'' to O_2 and Rot (O_2, β) keeps O_2 fixed. Therefore if Rot (O_2, β) Rot (O_1, α) is Rot (O, α + β) then $OO_1 = OO_1'$ and $OO_2'' = OO_2$. This tells us that O lies on the perpendicular bisectors of O_1O_1' and $O_2''O_2$. Let and d_1 and d_2 be the perpendicular besectors of O_1O_1' and O_2O_2'' respectively. Then d_2 makes α/2 with O_2O_1 and d_1 makes β/2 with O_2O_1, (Fig. 4.138). If we draw d_1, d_2 satisfying the above conditions then O is the desired centre of rotation. ❑

Problem 15. Construct equilateral triangles on the sides of a triangle ABC inwardly. Prove that the centres X, Y, Z of these triangles themselves form the vertices of an equilateral triangle.

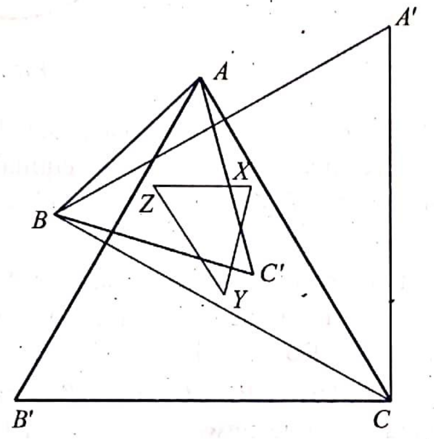


Fig. 4.139

Equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are described inwardly as in Fig. 4.139. Let X, Y, Z be the centres of the triangles $BA'C$; $CB'A$ and $AC'B$ respectively. Rotate through 120° about X, then rotate through 120° about Y and then rotate through 120° about Z. The first transformation Rot (X, 120°) takes B to C since $\angle BXC = 120°$; then Rot (Y, 120°) takes C to A and Rot (Z, 120°) takes A to B. Thus the sum of the three rotations leaves B fixed. But the sum of the three rotations Rot (X, 120°), Rot (Y, 120°) and Rot (Z, 120°), is a translation (Problem 14); and this translation leaves B fixed implies that this translation must be the identity transformation. Now Rot (X, 120°) followed by Rot (Y, 120°) is Rot (O, 240°) where O is the point of intersection of the lines through X and Y making angles 60° with XY (see Problem 14). This means that ΔXYO is equilateral. Now Rot (O, 240°) followed by Rot (Z, 120°) is the identity transformation. Hence O and Z should coincide. Thus $\angle XYZ$ is equilateral. ❑

Remark. The above proof also works for problem 13 where the equilateral triangles are drawn outwardly. The equilateral triangles XYZ formed by their centres are called the *outer Napoleon triangle* and *inner Napolean triangle* respectively.

Problem 16. If the perpendiculars from a point P of the circumcircle of ΔABC to the sides BC, CA, AB meet the circumcircle again at A_1', B_1', C_1' then AA_1', BB_1', CC_1' are all parallel to the Simson line of P with respect to ΔABC.

See Fig. 4.140. We have $\angle A_1'AC = \angle A_1'PC = \angle A_1PC = \angle A_1B_1C$ (since quadrilateral A_1CPB_1 is cyclic). Therefore $AA_1' \parallel C_1B_1A_1$. Similarly BB_1' CC_1' are also parallel to the Simson line $A_1B_1C_1$ of P. ❑

Problem 17. Find all the points on the circumcircle of a given triangle ABC whose Simson lines all have a given direction.

Draw BX parallel to the given direction meeting the circumcircle of ΔABC at X (Fig. 4.141). Draw the perpendicular from X to the side AC meeting the circle again at P. Then by the previous problem (Problem 16) P is the required point and the Simson line of P is $A_1B_1C_1$ parallel to BX through B_1 (Fig. 4.141). Again, problem 16 implies that P is the unique point whose Simson line is parallel to the given direction. ❑

Problem 18. Construct ΔABC given A, $b + c$ and h_a.

Suppose ΔABC is the required triangle. Let the internal bisector of $\angle A$ meet BC at U and the circumcircle at P. Drop the perpendiculars PC_1 and PB_1 on AB, AC respectively (Fig. 4.142). We have

$$AC_1 + AB_1 = AB + BC_1 + AC - B_1C = AB + AC + BC_1 - B_1C$$
$$= b + c + BC_1 - B_1C \qquad (*)$$

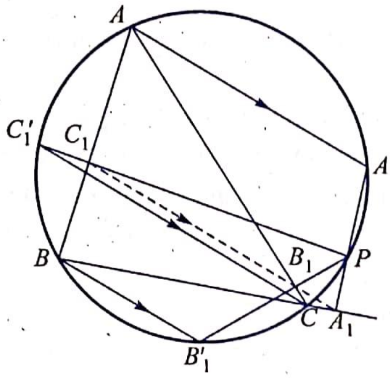


Fig. 4.140

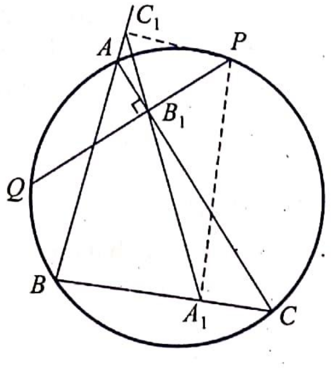


Fig. 4.141

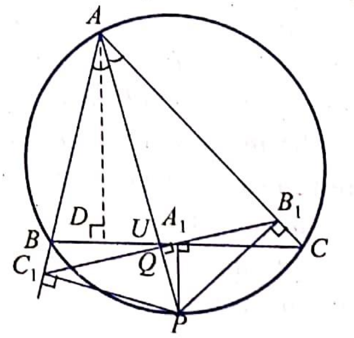


Fig. 4.142

We know that $PB = PC$ (why?) and $PC_1 = PB_1$ ($\because$ P is on the bisector of $\angle A$). Therefore $\Delta PAC_1 = \Delta PAB_1$ and $\Delta PC_1B = \Delta PB_1C$. Hence $AC_1 = AB_1$, $BC_1 = B_1C$. Substituting in (*) we get $AC_1 + AB_1 = b + c$ and therefore $AC_1 = AB_1 = (1/2)(b + c)$.

In the quadrilateral AC_1PB_1 we know $\angle B_1AC_1 = \angle A$, the sides AC_1, AB_1 and the other two angles $\angle AC_1P$ and $\angle AB_1P$ are right angles. This means that we can construct quadrilateral AC_1PB_1. Now AP is the perpendicular bisector of B_1C_1. We observe that B_1C_1 is the Simson line of P for ΔABC. Therefore B_1C_1 cuts BC at A_1 the foot of the perpendicular from P on BC. Suppose B_1C_1 meets AP at Q. Consider the triangles ADU and PQA_1.

$$\angle ADU = \angle PQA_1 = 90°, \angle DAU = \angle QPA_1$$

(alternate angles; $AD \parallel PA_1$)

Therefore $\Delta ADU \,|||\, \Delta PQA_1$. This gives

$$\frac{PQ}{AD} = \frac{PA_1}{AU} = \frac{PA_1}{AP - UP} \quad (1)$$

In the right triangle PA_1U, A_1Q is the altitude from A_1

Therefore $\quad PA_1^2 = PQ \,.\, PU \quad (2)$

Similarly from ΔPAB_1, $PB_1^2 = PQ \,.\, PA \quad (3)$

From (1), (2) and (3) we have $PQ \,.\, AU = PA_1 \,.\, AD = PA_1^2 = PQ \,.\, PU$

$$\therefore \quad PQ\,(AU + UP) = PA_1^2 + PA_1 \,.\, AD.$$

$$PQ\,(AP) = PA_1^2 + PA_1 \,.\, AD$$

$\therefore$ (3) gives $PA_1^2 + PA_1 \,.\, AD - PB_1^2 = 0 \quad (4)$

In this equation (4) we know $AD = h_a$, PB_1 is known from the quadrilateral AC_1PB_1. So, PA_1 may be constructed.

(Exercise: Construct two segments given their product q and their sum p. In other words solve geometrically, the quadratic equation $x^2 - px + q = 0$).

Now draw the circle with centre P and radius PA_1 meeting B_1C_1 at A_1. The line perpendicular to PA_1 through A_1 meets AC_1, AB_1 at B, C respectively. This gives ΔABC.

❑

Note. This problem has no solution if $h_a > AQ$ (depends only on $b + c$ and A). $h_a = AQ$ implies ΔABC is isosceles and $h_a < AQ$ gives two symmetric solutions with respect to AP.

Problem 19. Let P_1, P_2 be any two points on the circumcircle of ΔABC. Then the angle between the Simson lines of P_1 and P_2 is half the angular measure of arc P_1P_2.

Let the perpendiculars from P_1, P_2 to BC meet the circumcircle again at X, Y respectively. (Fig. 4.143). Then we know that the Simson line of P_1 is parallel to AX and the Simson line of P_2 is parallel to AY (Problem 16). Therefore the angle between the Simson lines of P_1 and P_2 is $\angle XAY$.

Now, $P_1X \parallel P_2Y$ implies that arc P_1P_2 = arc XY.

The required angle = $\angle XAY = (1/2)\,\angle XSY = (1/2)\,\angle P_1SP_2$. ❑

Remark. As an immediate consequence of Problem 19 we note that the perpendiculars through P_1, P_2 to the Simson lines of P_2, P_1 respectively meet at the circumcircle of ΔABC. The same is true for the parallels through P_1, P_2.

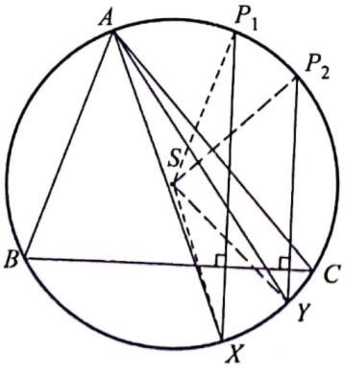


Fig. 4.143

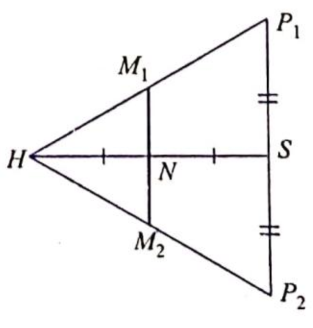


Fig. 4.144

Problem 20. If P_1 and P_2 are two diametrically opposite points on the circumcircle of ΔABC, then their Simson lines are perpendicular to each other and intersect on the nine-point circle of ΔABC.

From the previous problem, the angle between the Simson lines of two diametrically opposite points P_1, P_2 is $180°/2 = 90°$. By Theorem 63, the Simson lines of P_1 and P_2 bisect the segments HP_1 and HP_2 where H is the orthocentre of ΔABC. Let the midpoints of HP_1 and HP_2 be M_1 and M_2. (See Fig. 4.144.)

Now the nine-point centre N is the midpoint of SH and S is the midpoint of P_1P_2. Therefore N must be the midpoint of M_1M_2.

$$NM_1 = (1/2)\ SP_1 = \frac{R}{2} = NM_2 = \text{radius of the nine-point circle.}$$

M_1M_2 is a diameter of the nine-point circle of ΔABC.

Suppose the Simson lines of P_1 and P_2 meet at X. Then $\angle M_1XM_2 = 90°$ Therefore X lies on the nine-point circle of ΔABC.

Problem 21. Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles inscribed in the same circle. It P is a point on this circle, the angle between the Simson lines of P with respect to $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ is a constant.

P be any point on the circumcircle of the given triangles. Draw PX, PY perpendicular to A_1C_1 and A_2C_2 respectively meeting the circle again at X, Y (Fig. 4.145). Then the Simson lines of P with respect to the two triangles are parallel to B_1X and B_2Y (Problem 16). The angle between B_1X and B_2Y is given by

$$\begin{aligned}
&(1/2)\ (\text{arc } XY - \text{arc } B_2B_1)\\
&= \angle XPY - (1/2) \text{ arc } B_2B_1,\\
&= \angle C_1ZC_2 - (1/2) \text{ arc } B_2B_1 \qquad \text{(angle between the perpendiculars)}\\
&= \angle A_1C_2A_2 + \angle C_1A_1C_2 - (1/2) \text{ arc } B_2B_1\\
&= (1/2)\ (\text{arc } A_1A_2 + \text{arc } C_1C_2 - \text{arc } B_2B_1)\\
&= \text{constant, independent of } P.
\end{aligned}$$

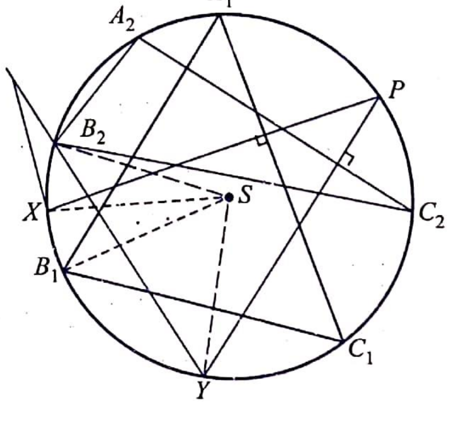


Fig. 4.145

Problem 22. Given a triangle ABC and two points P_1, P_2 on its circumcircle, there exists a third point P_3 on the circumcircle such that the Simson lines of P_1, P_2, P_3 are

concurrent. Further, the point of concurrence is the midpoint of the line segment joining the orthocentres of ΔABC and $\Delta P_1P_2P_1$. Also the Simson lines of A, B, C with respect to $\Delta P_1P_2P_3$ are concurrent at the same point. The Simson line of each of P_1, P_2, P_3 is perpendicular to the line joining the other two, Conversely, if a chord QR is perpendicular to the Simson line of some point P on the circumcircle of ΔABC, then the Simson lines of P, Q, R are concurrent.

In view of Problem 17, for any two distinct points P_1, P_2 on the circumcircle, the Simson lines are not parallel. Let them meet at point X.

If H is the orthocente of ΔABC, join HX and produce it to H' such that $HX = XH'$. Let P_3 be the orthocentre of $\Delta P_1P_2H'$. (See Fig. 4.146)

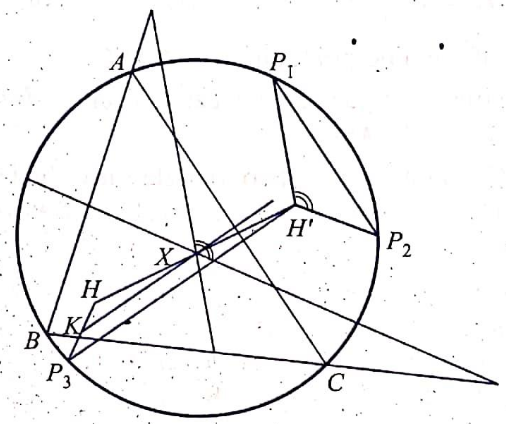


Fig. 4.146

The Simson line of P_1 bisects P_1H and by construction X is the midpoint of HH'. Therefore P_1H' is parallel to the Simson line of P_1. Similarly, P_2H' is parallel to the Simson line of P_2. So, $\angle P_2H'P_1$ = angle between the Simson lines of P_1 and P_2 (in our figure it is the obtuse angle).

$$\therefore \qquad P_2H'P_1 = 180° - \frac{1}{2}\text{arc } P_1P_2 = 180° - \angle P_1AP_2.$$

But $\angle P_1P_3P_2 = 180° - \angle P_2H'P_1 = \angle P_1AP_2$ and hence P_3 lies on the circumcircle of ΔABC.

Further the Simson line of P_3 must be parallel to P_3H' (why ?) passing through the midpoint K of P_3H. This means that the Simson line of P_3 has to pass through X. The point of concurrence X of the Simson lines of P_1, P_2, P_3 is the midpoint of the line segment joining the orthocentres H and H' of the triangles ΔABC and $\Delta P_1P_2P_3$ respectively. The Simson line of P_1 is parallel to P_1H' and hence perpendicular to P_2P_3. By symmetry, we see that the Simson line of each of P_1, P_2, P_3 is perpendicular to the line joining the other two.

Conversely suppose a chord QR is perpendicular to the Simson line of some point P on the circumcircle of ΔABC, then the Simson lines of P, Q, R are concurrent. For, if the simson lines of P and Q intersect at X, extending HX to H' such that $HX = XH'$ we see that PH' is parallel to the Simson line of P with respect to ΔABC.

Therefore $PH' \perp QR$. By the first part of the problem H' is the orthocentre of ΔPQR_1 where R_1 is that point whose Simson line with respect to ΔABC passes through X. This means that QR_1 is also perpendicular to PH'.

Therefore $R = R_1$. Thus the Simson lines of P, Q, R with respect to ΔABC are concurrent. Further it is clear from our discussions that the Simson line of A, B, C with respect to $\Delta P_1P_2P_3$, also concur at the same point X. ❑

Problem 23. PQ is a chord of a circle. Through the midpoint M of PQ chords AB and CD are drawn. AD and BC meet PQ at K and L. Then prove that M is the midpoint of KL (*Butterfly theorem*).

Draw perpendiculars KX_1, LY_1 from K, L on AB, CD respectively (Fig. 4.147). Similarly draw KX_2, LY_2 perpendicular to CD, AB from K and L.

$$\Delta MKX_1 \ ||| \ \Delta MLY_2 \text{ gives } \frac{MK}{ML} = \frac{KX_1}{LY_2} \quad (1)$$

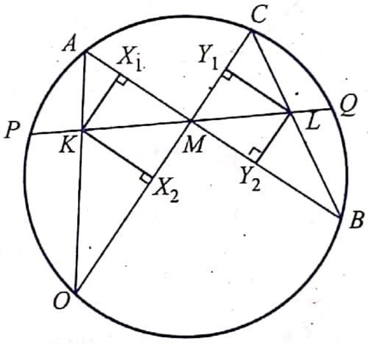


Fig. 4.147

$$\Delta MKX_2 \ ||| \ \Delta MLY_1 \text{ gives } \frac{MK}{ML} = \frac{KX_2}{LY_1} \quad (2)$$

$$\Delta AKX_1 \ ||| \ \Delta CLY_2 \text{ gives } \frac{KX_1}{LY_1} = \frac{AK}{CL} \quad (3)$$

$$\Delta DKX_2 \ ||| \ \Delta BLY_2 \text{ gives } \frac{KX_2}{LY_2} = \frac{DK}{BL} \quad (4)$$

$$\therefore \left(\frac{MK}{ML^2}\right) = \frac{KX_1}{LY_2} \cdot \frac{KX_2}{LY_1} = \frac{AK \cdot DK}{CL \cdot BL} \quad \text{from (1), (2), (3) and (4).}$$

$$= \frac{PK \cdot KQ}{PL \cdot LQ} = \frac{(PM - KM)(MQ + KM)}{(PM + ML)(QM - ML)}$$

$$= \frac{PM^2 - MK^2}{PM^2 - ML^2} \quad \text{(Since } PM = MQ\text{)}$$

Now $$\frac{MK^2}{ML^2} = \frac{PM^2 - MK^2}{PM^2 - ML^2} \quad \text{hence } MK = ML$$ ❑

Problem 24. Let the incircle touch the side BC of ΔABC at X. If A' is the midpoint of BC then prove that $A'I$ bisects AX.

Let K' be the diametrically opposite point to X on the incircle of ΔABC. Draw B_1C_1 tangent to the incircle as in Fig. 4.148. Then $B_1C_1 \parallel BC$ and $\Delta AB_1C_1 \ ||| \ \Delta ABC$. The incircle of ΔAB_1C_1 should touch B_1C_1 at X_1 the intersections of AX and B_1C_1. (Why?).

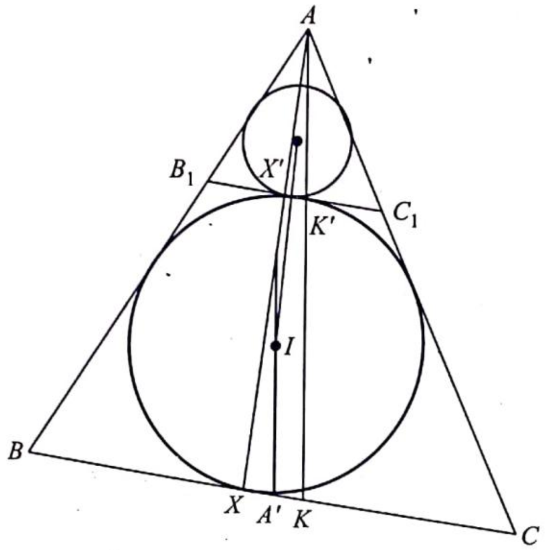


Fig. 4.148

We have $YY_1 = B_1Y_1 + B_1Y = B_1X_1 + B_1K' = 2B_1X_1 + X_1K'$

or $B_1X_1 = (1/2)\,(YY_1 - X_1K_1) = (1/2)\,(ZZ_1 - X_1K') = C_1K'$.

Similarly $BX = KC$ (Fig. 4.148).

Hence A' is the midpoint of XK as well. This means that $A'I$ should be parallel to KK' which implies that $A'I$ bisects XA. ❑

Problem 25 (Morley's theorem). The points of intersection of the adjacent trisectors of the angles of any triangle form the vertices of an equilateral triangle.

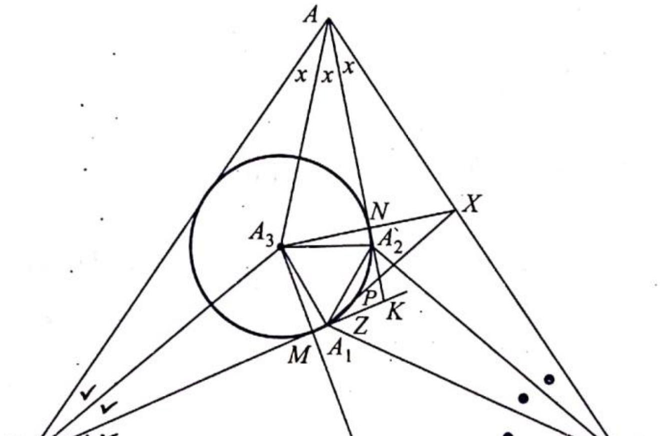


Fig. 4.149

Let the trisectors BA_1 and AA_2 meet at K (See Fig. 4.149). For the triangle ABK, the incentre is A_3. Let the incircle of ΔABK touch BA_1 and AA_2 at M, N respectively. Suppose A_3N meets AC at X and A_3M meets BC at Y. Let the tangent from X to the incircle touch the circle at P. We have $A_3N = NX$ as AN bisects $\angle A_3AX$ and $AN \perp A_3X$. So, $A_3P = A_3N = (1/2)\,A_3X$. Also $\angle A_3PX = 90°$. This means that $\angle PA_3X = 60°$ and $\angle A_3XP = 30°$ (1)

Further $\angle MA_3N = 180° - \angle MKN$ (as quadrilateral $MKNA_3$ is cyclic).

$= 180° - (180° - (2/3)(\angle B + \angle A))$

$= (2/3)(\angle B + \angle A) = (2/3)(180° - \angle C)$

$= 120° - (2/3)(\angle C)$ (2)

Suppose the tangent XP from X to the incircle of ΔABK meets BK at Z.

Then $\angle ZYM = \angle ZA_3M$ (since ΔZA_3Y is isosceles, as $A_3M = MY$, $MZ \perp A_3Y$).

$= (1/2)\angle MA_3P$ (since tangents at P and M meet at Z)

$= (1/2)(\angle MA_3P - \angle PA_3X) = (1/2)(\angle MA_3N - 60°)$

$= (1/2)(120° - (2/3)\angle C - 60°)$ from (2)

Thus $\angle ZYM = 30° - (1/3)\angle C$ (3)

$\angle A_3YX = \angle A_3XY$ (as $A_3X = A_3Y = 2A_3M$)

$= (1/2)(180° - \angle XA_3Y)$

$= (1/2)\angle MKN$ (since quadrilateral $MKNA_3$ is cyclic)

$= (1/2)(180° - (2/3)(\angle B + \angle A)$

$= (1/2)(180° - (2/3)(180° - \angle C)$

$= (1/2)(60° + (2/3)\angle C) = 30° + (1/3)\angle C.$

$\angle ZYX = A_3YX - \angle A_3YZ = \angle A_3YX - \angle MYZ$

$$= \left(30° + \frac{\angle C}{3}\right) - \left(30° - \frac{\angle C}{3}\right) = \frac{2}{3}\angle C$$

and $\angle ZXY = \angle A_3XY - \angle A_3XZ$

$$= \left(30° + \frac{\angle C}{3}\right) - 30° = \frac{\angle C}{3} \quad \text{from (1)}$$

$\therefore$ $\angle XZY = 180° - \angle C$ or $ZYCK$ is cyclic.

This means that Z coincides with A_1 and the tangent from X to the incircle passes through A_1. Similarly, the tangent from Y passes through A_2. By symmetry $\angle A_1A_3P = \angle A_2A_3N$; $\angle PA_3X = 60°$ implies that $\angle A_1A_3A_2 = 60°$. Again by symmetry each angle of $\Delta A_1A_2A_3$ is equal to 60°. ❑

PROBLEMS

1. AOB is a given angle. Two circle of a radius r_1, r_2 touch OA, OB and also touch each other. Find the radius of another circle touching the sides of $\angle AOB$ and having its centre at the point of contact of the two given circles.
2. S_1 and S_2 are two non-intersecting circles with their centres at a distant 'd' apart. Prove that the four points of intersection of direct common tangents and transverse common tangents are concylic. Find the radius of this circle.
3. S is a circle of radius R and centre O. Two other circles of radii r_1, r_2 touch S internally and they intersect at A, B. Find $r_1 + r_2$.
4. Given two circles, find the locus of point P such that the ratio of the lengths of the tangents drawn from P to the given circles is a constant.
5. Given three pairwise intersecting circles, prove that the three common chords of the circles are concurrent.

6. A point A is chosen inside a circle. Find the locus of the point of intersection of the tangents drawn to the circle at the extremities of chords passing through A.
7. A straight line meets AB, BC and AC produced at D, E and F respectively. Prove that the midpoints of DC, AE and BF, are collinear.
8. Two circles touch each other internally at A. A tangent to the smaller circle meets the larger circle at B, C. Find the locus of the incentre of ΔABC.
9. Points P, Q, R are taken on BC, CA, AB such that $BP/PC = CQ/QA = AR/RB = m/n$
   Prove that area $\Delta PQR = (m^3 + n^3)/(m + n)^3$ area ΔABC.
10. Construct ΔABC given m_b, m_c, h_a.
11. Construct a triangle given a, m_b/m_c, $b^2 - c^2$.
12. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle on the four bisectors of the other two angles are collinear.
13. Given the six midpoints of the arcs subtended by the sides of a given ΔABC. Prove that one can construct I, I_a, I_b, I_c with ruler alone.
14. Construct ΔABC given r, r_a, m_a.
15. Given the positions of I, I_a and the length h_a as well as r_a/r, construct ΔABC.
16. Construct ΔABC given $b - c$, B, r.
17. Construct ΔABC given $b - c$, h_b, r.
18. Construct ΔABC, given A, X, X_a (notations as in the text).
19. Four points A, B, C, D such that each is the orthocentre of the triangle formed by the other three are said to form an orthocentric group of points. Prove that
    (*i*) the four triangles of an orthocentric group have the same orthic triangle.
    (*ii*) the four triangles of an orthocentric group have the same nine-point circle.
    (*iii*) the circumradii of the four triangles of an orthocentric group are equal.
    (*iv*) the circumcentres of an orthocentric group of triangles form an orthocentric quadrilateral.
    (*v*) An orthocentric group of triangles and the orthocentric group of their circumcentres have the same nine-point circle.
    (*vi*) The four vertices of a given orthocentric group of triangles may be considered as the circumcentres of a second orthocentric group of triangles.
    (*vii*) The four centroids of an orthocentric group of triangles form an orthocentric group.
    (*viii*) The nine-point centre of an orthocentric group of triangles is the same as the nine-point centre of the orthocentric group formed by the centroids of the given group of triangles.
    (*ix*) In any ΔABC, prove that I, I_a, I_b, I_c form an orthocentric group.
    (*x*) Δ is the orthic triangle of the orthocentric group I, I_a, I_b, I_c and the circumcircle of ΔABC is the nine-point-circle of the same orthocentric group.
    (*xi*) The circumcentres of the group I, I_a, I_b, I_c are the incentre and excentres of the symmetric of ΔABC with respect to the circumcentre S of ΔABC.
    (*xii*) Show that the algebraic sum of the distance of the points of an orthocentric group from a straight line passing through the nine-point centre of the group is equal to zero.
20. If h, m, t the altitude, the median, and the internal bisector from the same vertex of a triangle then prove that $4R^2h^2(t^2 - h^2) = t^4(m^2 - h^2)$ where R is the circumradius of the triangle.
21. $ABCD$ is a cyclic quadrilateral. Prove that the perpendiculars from the midpoints of AB, BC, CD, DA to CD, DA, AB, BC respectively concur at a point.

22. $ABCD$ is a cyclic quadrilateral; H_1, H_2, H_3, H_4 are the orthocentres of ΔBCD, ΔCDA, ΔDAB and ΔABC respectively. Prove that AH_1, BH_2, CH_3, DH_4 bisect each other.

23. If X is the common point to AH_1, BH_2, CH_3, DH_4 in problem 22, prove that the nine-point circles of ΔABC, ΔBCD, ΔCDA, ΔDAB pass through X.

24. With notations as in problems 22, 23 prove that $XA^2 + XB^2 + XC^2 + XD^2 = 4R^2$ where R is the radius of the circle $ABCD$.

25. Prove that the incentres of ΔABC, ΔBCD, ΔCDA, ΔDAB in a cyclic quadrilateral $ABCD$ form a rectangle.

26. For ΔABC, we call I, I_a, I_b, I_c as the tritangent centres. Prove that the sixteen tritangent centres, of the four triangles ABC, BCD, CDA, DAB of a cyclic quadrilateral lie by fours on eight straight lines; these eight lines consist of two perpendicular groups of four parallel lines.

27. If a, b, c, d are the sides of a cyclic quadrilateral $ABCD$, prove that (area $ABCD)^2 = (s-a)(s-b)(s-c)(s-d)$ where $2s = a + b + c + d$.

28. $ABCD$ is a cyclic quadrilateral in which $AC \perp BD$. Prove

(*i*) The midpoints of the sides of the quadrilateral ABCD are concyclic and their centre is the centroid of $ABCD$.

(*ii*) If AC, BD meet at O the perpendicular from O to AB bisects CD.

(*iii*) If X, Y, Z, W are the feet of the perpendiculars from O on the sides of the quadrilateral, X, Y, Z, W lie on the circle passing through the midpoints of the sides.

(*iv*) If S is the circumcentre of $ABCD$, then the distance of S from AB is equal to $CD/2$.

(*v*) $AB^2 + CD^2 = BC^2 + AD^2 = AR^2$ where R is the circumradius of $ABCD$.

29. Prove that in any triangle ABC, $b^2 - c^2 = 2a\,A'D$ where D the foot of the altitude from A and A' is the midpoint of AB.

30. If P is a point on the side BC of ΔABC such that $BP/PC = m/n$, then prove that $mb^2 + nc^2 = (m + n)AP^2 + mPC^2 + nPB^2$.

31. If three equal circles have a common point then prove that the circle through the other three intersections is equal to them.

32. In problem 31, if the centres of the three equal circles are C_1, C_2, C_3 and their points of intersection are O, A, B, C prove that the figure formed by O, A, B, C is congruent to the figure formed by C, C_1, C_2, C_3 where C is the centre of circle ABC.

33. In problem 32 prove that in either of these congruent figures, the line joining any two of the vertices is perpendicular to the line joining the other two.

34. On a line segment AB a semicircle is drawn. On the other side of AB: a rectangle $ABDC$ is drawn with AC equal to the side of the square inscribed in the circle. P is any point on the semicircle; PC, PD cut AB at X, Y. Prove that $AX^2 + BF^2 = AB^2$.

35. ABC is an isosceles triangle with $AB = AC$; P, Q are points on AB : S_1 is the circle (P, PB) and S_2 is the circle (Q, QB). Similarly R, S are points on AC; S_3 is the circle (R, RC) and S_4 is the circle (S, SC). Let S_1 and S_3 meet at X, Y and S_2 and S_4 meet at Z, W. If PR and QS meet at T, prove that T is the centre of the circle passing through X, Y, Z, W. If $PR \parallel QS$, then prove that X, Y, Z, W lie on a straight line perpendicular to PR.

36. XY is a chord of a circle and P is the midpoint of XY. AB and CD are chords through P. Prove that AC and BD cut XY at equal distances from P, the same being true for AD, BC.

37. If S is a circle and AX, BY, CZ are the tangents from A, B, C to S, then prove that the circles of ΔABC touches S if and only if $AB \,.\, CS \pm AC \,.\, BY \pm BC \,.\, AX = 0$.

38. A, C, B are three colliner points in that order. Consider the semicircles on AB, BC, CA on the same sides of AB. The figure bounded by the above semicircles is called **'Shoemaker's Knife'**. Let the perpendicular through C to AB meet the semicircle on AB at D. Let TU be

the direct common tangent to the circles on AC and BC touching these circles at T, U respectively. Let TU meet CD at X. Prove that

1. arc ADC = arc ATC + arc CUB
2. $DC^2 = TU^2 = AC \,.\, BC$
3. X is the centre of the circle through C, D, T, U.
4. There area of **'Shoemaker's Knife'** equals the area of the circle having CD as diameter
5. AD passes through T and BD passes through U.
6. If S_1 and S_2 are the circles inscribed in the curvilinear triangles ACD, BCD prove that S_1 and S_2 are equal circles and diameter of S_1 equals $AC \,.\, BC/AB$.
7. If the circle S_1 touches the arc AC at M then the common tangent to the two circles at M passes through B.
8. Prove that the smallest circle tangent to an circumscribing S_1 and S_2 is the circle on CD as diameter.

39. D, E, F are points on the sides BC, CA, AB of ΔABC. Prove that the circles AEF, BDF and CDE meet at a point.

40. If we call the common point of the three circles in prob. 39 as M, prove that MD, ME, MF make equal angles with the respective sides.

Further prove that $\angle BMC = \angle BAC + \angle EDC$.

41. Prove that the circumcircles of the four triangles formed by four lines have a common point.

42. Given four lines in a plane, prove that there is one and only one point from which the feet of the perpendiculars to the lines are collinear. What is that point? (Hint: problem 41).

43. ABC is a triangle and D, E, F are points on BC, CA, AB. By problem 39, the circles AEF, BFD and CDE have a common point M. If AP, BP, CP are three concurrent lines meeting the circles AEF, BFE, CDE at X, Y, Z respectively prove that X, Y, Z, M, P all lie on a circle.

44. Let F, F_a, F_b, F_c be the points of contact of the nine-point circle with the incircle and the three exscribed circles of ΔABC, prove that the point of intersection of the diagonals of the quadrilateral formed by F, F_a, F_b, F_c lies on its midline.

45. With notations as in Prob. 44, if the internal and external bisectors of angles A, B, C meet BC, AB at A_1, B_2, C_1, C_2 respectively,

prove that

$$\Delta F_aF_bF_c \equiv \Delta A_1B_1C_1$$
$$\Delta FF_bF_c \equiv \Delta A_1B_2C_2$$
$$\Delta FF_cF_a \equiv \Delta B_1C_2A_2$$
$$\Delta FF_aF_b \equiv \Delta C_1A_2B_2$$

46. $ABCD$ is a cyclic quadrilateral. Four circles α, β, γ, δ touch the circle $ABCD$ at A, B, C, D respectively. Let $t_{\alpha\beta}$ be the segment of the direct common tangent to α, β if α, β touch the circle $ABCD$ in the same manner (both internally or both externally); let $t_{\alpha\beta}$ be the segment of the transverse common tangent if α, β touch the given circle in different ways. Similarly we define, $t_{\beta\gamma}$ etc. Prove that

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta}.$$

47. On the circle K there are three distinct points A, B, C. Using a straight edge and a compass, construct a fourth point D on K such that a circle can be inscribed in the quadrilateral $ABCD$.

48. A circle is inscribed in ΔABC. Tangents to the circle parallel to the sides are constructed. Each of these tangents cuts off a triangle from ΔABC. In each of these triangles a circles is inscribed. Find the sum of the areas of all the four inscribed circles.

49. Let $A_0B_0C_0$ and $A_1B_1C_1$ be two acute angled triangles. Let $f = \{\Delta ABC \mid \Delta ABC \parallel\!| \Delta A_1B_1C_1, A_0$ is on BC, B_0 is on CA and C_0 is on $AB\}$. Find a triangle in f of maximum area and construct it.

50. Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

51. P is any point inside ΔABC; u, v, w be the distances from P to the vertices A, B, C respectively; x, y, z be the distances from the point P to the sides BC, CA, AB respectively. Then prove that

(*i*) $au + bv + cw \geq 4\Delta$

(*ii*) $u + v + w \geq 2(x + y + z)$

(*iii*) $ux + vy + wz \geq 2(xy + yz + zx)$

(*iv*) $2(l/u + l/v + l/w) \leq l/x + l/y + l/z$

(*v*) $uvw \geq R(x + y)(y + z)(z + x)/zr$

(*vi*) $uvw \geq 4Rxyz/r$

(*vii*) $uv + vw + wu \geq 2R(xy + yz + zx)/r$

(*viii*) $u + v + w \geq 6r$

52. Prove the following for a triangle ABC.

(*i*) $3(bc + ca + ab) \leq (a + b + c)^2 \leq 4(bc + ca + ab)$

(*ii*) $a^2 + b^2 + c^2 \geq 36(s^2 + abc/s)35.$

(*iii*) $8(s - a)(s - b)(s - c) \leq abc.$

(*iv*) $abc < a^2(s - a) + b^2(s - b) + c^2(s - c) \leq 3abc/2.$

(*v*) $bc(b + c) + ca(c + a) + ab(a + b) \geq 48(s - a)(s - b)(s - c).$

(*vi*) $2s/abc \leq 1/a^2 + 1/b^2 + 1/c^2.$

(*vii*) $3/2 \leq a/(b + c) + b/(c + a) + c/(a + b) < 2.$

(*viii*) The perimeter of the triangle $\leq s$.

53. Prove that ΔABC is acute, right or obtuse according as $a^2 + b^2 + c^2 - 8R^2$ is positive, zero or negative.

54. If in ΔABC, $a^2 + b^2 > 5c^2$, then show that c is the smallest side.

55. If p is the perimeter of the triangle whose vertices are the points of contact of the incircle with the sides of ΔABC, prove that $p \geq 6r^3 \sqrt{s/4R}$.

56. ABC is a right triangle with $\angle A = 1$ radian and right angled at C. If I is the incentre and O is the midpoint of AB and N is the midpoint of OC, find whether ΔNIO is an acute, obtuse or right triangle.

57. Find a necessary and sufficient condition on a quadrilateral $ABCD$ in order that there exists a point P in the plane of $ABCD$, such that the areas of the triangles PAB, PBC, PCD, PDA are all equal.

58. ABC is a triangle such that a, b, c are all unequal; G, I, H are the centroid, incentre and orthocentre of ΔABC. Prove that $\angle GIH > 90°$.

59. Let ABC be a triangle; K, L be points on BC trisecting BC; M, N be on CA trisecting CA; P, Q be on AB trisecting AB. Construct equilateral triangles KLA; MNB', PQC' all directed inwards. Show that $A'B'C$ is an equilateral triangle having the same centroid as ΔABC.

60. In ΔABC, the internal bisector of $\angle A$ and the median through A meet BC in two distinct points L and D. The perpendicular from B upon AL meets AL and AD in P and M respectively. The perpendicular from C upon AL meets AL and AD in Q and N respectively. Show that $LM \parallel AB$ and $LN \parallel AC$.

61. Let D be a variable point on the side BC of ΔABC. Suppose the direct common tangent to the incircles of ΔABD and ΔACD, other than BC, meets AD in E, find the locus of E.

62. ABC is a mangle with $\angle A = 30°$. S is the circumcentre and I is the incentre of ΔABC. D is a point on segment AB and E is a point on segment CA such that $BD = CE = BC$. Show that $SI \perp DE$ and $SI = DE$.

63. $ABCD$ is a convex quadrilateral in which $AC = BD$; XAB, YBC, ZCD and WDA are equilateral triangles and S_1, S_2, S_3, S_4 are their centres respectively. Prove that $S_1 S_3 \perp S_2 S_4$.

64. S_1 and S_2 are two circles which touch externally at P; S is a circle which touches S_1 and S_2 internally. A direct common tangent to S_1, S_2 meets S at B, C. The common tangent to S_1, S_2 at P meets S at A such that P, A lie on the same side of BC. Prove that P is the incentre of ΔABC.

65. ABC is a triangle and P is any point inside ΔABC; X, Y are the feet of the perpendiculars from P to AB, AC; Z, W are the feet of the perpendiculars from A to BP, CP. Prove that the lines ZY, WX and BC are concurrent.

66. ΔABC is an acute angled triangle; A' is the midpoint of BC and P is any point on the median AA' such that $PA' = BA'$. The perpendicular from P to BC cuts BC at X. The perpendicular from X to PB cuts AB at Z and the perpendicular from X to PC cuts AC at Y. Show that the circle XYZ touches BC at X.

67. I is the incentre of ΔABC; C' is the midpoint of AB and B' is the midpoint of CA. The line $C'I$ meets AC at B_2 and the line $B'1$ meets AB at C_2. If area of ΔAB_2C_2 equals area of ΔABC, find ΔBAC.

68. AB and CD are chords of a circle cutting each other at E; M is a point on the chord AB such that $AM/AB = m/n$. The tangent at E to the circle DEM cuts BC at X and CA at Y. Prove that $YE/EX = m/(n - m)$.

69. In ΔABC, BD and CE are the bisectors of $\angle B$, $\angle C$ cutting CA, AB at D, E respectively. If $\angle BDE = 24°$ and $\angle CED = 18°$, find the angles of ΔABC.

70. $A_0B_0C_0$ is a triangle and P is inside it; A_1, B_1; C_1 are the feet of the perpendiculars from P on the sides of $\Delta A_0 B_0 C_0$; A_2, B_2, C_2 are the feet of the perpendiculars from P on the sides of $\Delta A_1B_1C_1$. Likewise we define $\Delta A_nB_nC_n$ for $n > 2$. Is one of $\Delta A_nB_nC_n$ for $n \geq 1$, similar to $\Delta A_0B_0C_0$?

71. $C_1, C_2, \ldots C_n$ be a sequence of circles inscribed in **Shoemaker's Knife** such that C_n touches S_1, S_2 and C_{n-1}, where S_1, S_2 are seimicircles on AB, AC respectively and C_0 is the semicircle on CB (See Problem 38). Prove that the distance of the centre of C_n from AB is n times the diameter of C_n.

72. S_1 and S_2 are two circles of unit radius Touching at point P; I is a common tangent to S_1, S_2 touching them at X, Y; C_1 is the circle touching S_1, S_2 and l. C_n is the circle touching S_1, S_2 and l. C_{n-1} for $n > 1$. By computing the diameters of C_n, prove that $1/1.2 + 1/2.3 + 1/3.4 + \ldots + 1/n(n + 1) + \ldots = 1$.

73. ABC is an isosceles triangle; l is a straight line passing through a vertex of ΔABC, dividing ΔABC into isosceles triangles. Find all such isosceles triangles.

74. Find all convex polygons, for which one angle is bigger than the sum of the remaining angles.

75. Let P be a set of finitely many points in the plane, not all in a straight line. Prove that there exists a straight line in die plane containing exactly two points of P.

76. Let P be a finite set of points in a plane, no three of which are collinear and not all in a circle. Prove that there is a circle in the plane containing exactly three points of P.

-

CHAPTER

5

QUADRATIC EQUATIONS AND EXPRESSIONS

5.1 INTRODUCTION

In this chapter we shall discuss equations of the form

$$y = ax^2 + bx + c \quad (1)$$

where a, b, c are real numbers. These are called *quadratic equations* in the variable x. To start with, we take a simple linear equation of the form

$$y = ax + b.$$

In particular, consider the equation

$$2x - 1 = 0$$

We know that the solution is $x = \frac{1}{2}$. Suppose on the other hand we draw the graph of the function

$$y = 2x - 1$$

on a coordinate plane. The plotting of the curve is done, as usual, by fixing values of x at reasonable intervals and working out the corresponding values of y.

x	0	1	– 1	2	– 2	...
$y = 2x - 1$	– 1	1	– 3	3	– 5	...

Therefore, (0, – 1), (1, 1), (– 1, – 3), (2, 3) and (– 2, – 5) are all points on the graph of the function $y = 2x - 1$. A little experimentation will show that any three of these points are collinear. In fact the graph of $y = 2x - 1$ will be a straight line. That this will be so for any equation of the form $y = ax + b$ will be proved in Chapter 7. It is therefore only necessary to plot two points on the graph of $y = 2x - 1$ and join them by a straight line, on the coordinate plane.

Let us now plot the straight line $y = 2x - 1$. Two points on this line may be taken as (0, – 1) and (1, 1). Fig. 5.1 shows the straight line joining them. Such a graphical representation of the function $y = f(x) = 2x - 1$ gives us a sure method of estimating the solution of the equation

$$2x - 1 = 0.$$

We have only to look for points on the straight line $y = 2x - 1$, for which $y = 0$. This happens somewhere between $x = 0$ and $x = 1$. The actual value of the root of the equation is $x = 1/2$. This is confirmed by the graph of the function. But it is important

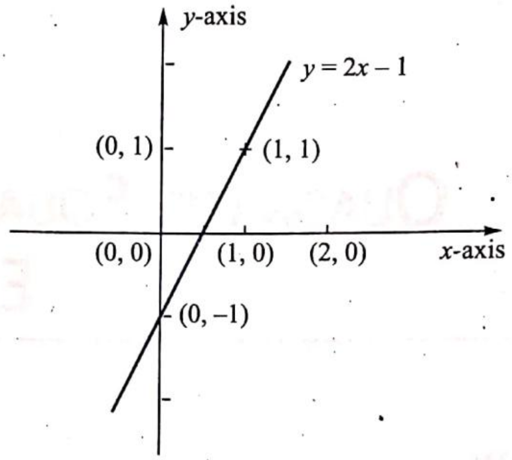


Fig. 5.1

to note that the graphical solution cannot be a precise substitute for the algebraic solution. In this case it is, because the answer is $x = 1/2$. But if the answer happens to be a more complicated number, the geometrical (that is graphical) solution may not give an accurate answer. This has nothing to do with the efficacy of the geometrical method. It is only a reflection or our equipment capabilities, like reading of measurements from a figure or a drawing.

To sum up, whenever we want to solve an equation of the form

$$ax + b = c \tag{2}$$

we may also do it geometrically by drawing the straight line graph

$$y = ax + b,$$

and looking for the x-coordinate of the point where the line $y = c$ (parallel to x-axis) cuts the line

$$y = ax + b.$$

For instance, to solve

$$2x + 3 = 5,$$

draw the graph of

$$y = 2x + 3.$$

Draw the line

$$y = 5.$$

The two meet at (1, 5). See Fig. 5.2. So $x = 1$ is the answer.

Now let us take a second degree expression, the simplest of which is $y = x^2$. Since the square of a real number is always non-negative, the graph lies above the x-axis. Moreover $x^2 = (-x)^2$ so that at both x and $-x$, y takes the same value. Thus the graph is symmetric about the y-axis. However, in the case of a first degree equation the graph being a straight line, can be completely drawn, once we plot any two points on the graph. The same cannot be done here. We can draw the graph of

$$y = x^2$$

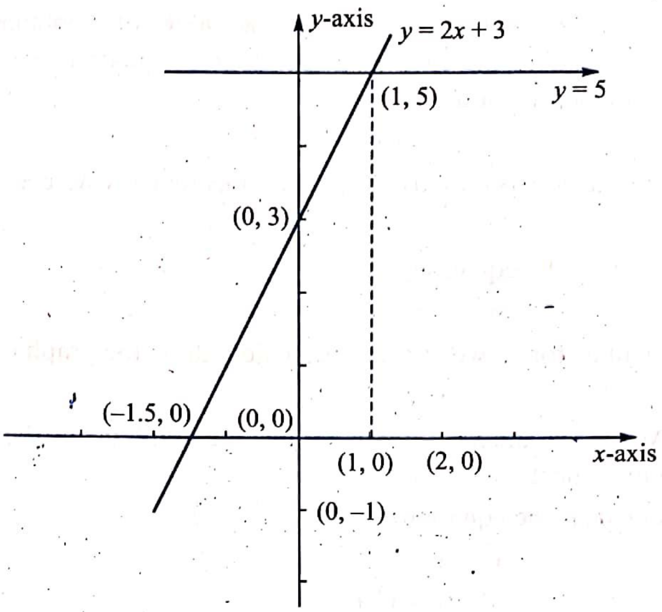


Fig. 5.2

only approximately On the coordinate plane using values of y at some chosen values of x. (See Fig. 5.3).

x	0	1	– 1	2	– 2	3	– 3	...		
$y = x^2$	0	1	1	4	4	9	9	...		

We are now ready to take the graph of the second degree expression, in the general case, viz.,

$$y = ax^2 + bx + c.$$

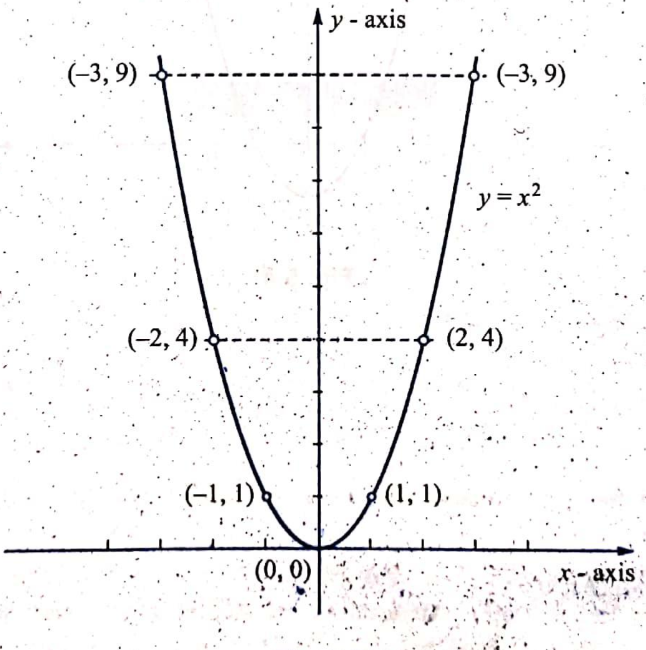


Fig. 5.3

The graph can be approximately drawn using the values of y obtained by taking x at reasonable intervals. As in the case of a first degree equation, we can look for the x-coordinates of the points on the graph of

$$y = ax^2 + bx + c$$

where it meets the x-axis. Since $y = 0$ on the x-axis, theoretically we get all the solutions of

$$ax^2 + bx + c = 0.$$

For example, consider the equation

$$x^2 - 1 = 0. \tag{3}$$

Giving various values for x, we can approximately draw the graph of

$$y = x^2 - 1$$

as in Fig. 5.4. At $x = + 1$ and $x = - 1$, we see that the graph meets the x-axis and at no other points do they meet.

Let us now consider the equation

$$x^2 + 1 = 0. \tag{4}$$

The graph of $y = x^2 + 1$

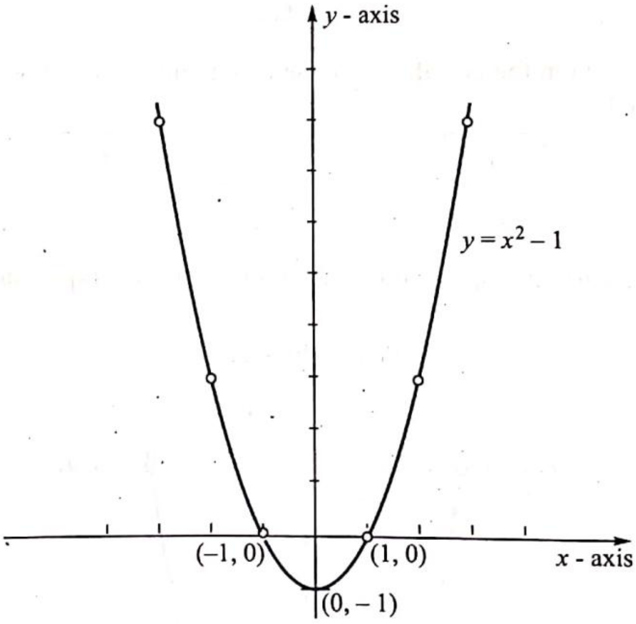


Fig. 5.4

is shown in Fig. 5.5.

x	0	1	– 1	2	– 2	3	– 3	...
$y = x^2 + 1$	1	2	2	5	5	10	10	...

We observe that the graph does not meet the x-axis at all. We conclude that the equation (4) has no solution in real numbers.

These three equations, viz., equations (2), (3) and (4) exhibit an important feature of equations of second degree. An equation of second degree may have two solutions, may have one solution, or may not have any solution, in real numbers. In any case it has at most two solutions.

If f is a real valued function defined on **R**, we can imitate the previous procedures to draw the graph of $y = f(x)$. Generally, it may be very difficult to draw even an approximate graph of $f(x)$. If it is possible to draw the graph of $f(x)$, many important properties of the function $f(x)$ which are not obvious from the algebraic form of $f(x)$ can be inferred by looking at its graph. Again the natural question is to solve the equation $f(x) = 0$. Any real or complex number α such that $f(\alpha) = 0$ is called a *zero of* $f(x)$. Theoretically, we can find all the zeros of $f(x)$ by looking for the x-coordinates of the points where the graph of $f(x)$ meets the x-axis. We emphasize again that drawing the graph of $f(x)$ may not be feasible in all cases even if the explicit algebraic form of $f(x)$ is known.

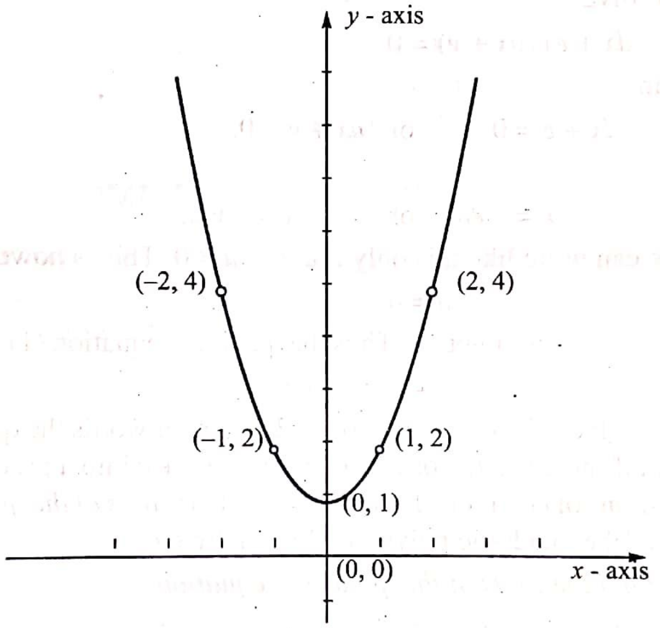


Fig. 5.5

EXERCISE 5.1

Draw the graphs of the following functions

1. $3x + 5$
2. $x - 6$
3. $x^2 + 4$
4. $2x^2 - 1$
5. $x^2 - 2x + 1$
6. $3x^2 - 2x + 5$
7. $x^3 - 1$

5.2 SOLUTION OF QUADRATIC EQUATIONS BY FACTORIZATION

In section 5.1, we observed that in general the graph of

$$y = ax^2 + bx + c, \quad a \neq 0$$

meets the x-axis at two points. This graphical method gives us a way of solving the equation

$$ax^2 + bx + c = 0, \quad a \neq 0 \tag{1}$$

provided the graph meets the x-axis. However, drawing the graph of such a function may not be a feasible task. We can only draw an approximate graph in general and can get only approximate solutions. An equation of the form (1) is called a quadratic equation. Because of this drawback of the graphical method, we look for algebraic methods of solving equation (1).

Suppose we can find numbers d, e, u and v such that

$$(ax^2 + bx + c) = (dx + e)(ux + v) \tag{2}$$

for all values of x. Then, to solve

$$ax^2 + bx + c = 0,$$

we have only to solve

$$(dx + e)(ux + v) = 0.$$

This would mean

either $\quad dx + e = 0 \quad$ or $\quad ux + v = 0.$

This gives

either $\quad x = -e/d \quad$ or $\quad x = -v/u.$

But note that we can write like this only if $d \neq 0$, $u \neq 0$. This is however true, since

$$du = a \qquad \text{from (2)}$$

and $a \neq 0$ by our starting assumption. Thus the quadratic equation (1) has two solutions

$$\alpha = -e/d,\ \beta = -v/u$$

whenever there is a factorization of the form (2). In other words the quadratic equation (1) can be solved if there is a factorization of $ax^2 + bx + c$ into a product of two linear factors. The solutions of equation (1) are also called the *roots of the quadratic equation* (1) or the zeros of the quadratic polynomial $ax^2 + bx + c$.

EXAMPLE 1. *Find the roots of the quadratic equation*

$$12x^2 + 25x + 12 = 0.$$

SOLUTION. Since $\quad 12 \times 12 = 144 = 16 \times 9 \quad$ and $\quad 25 = 16 + 9,$

we have

$$\begin{aligned} 12x^2 + 25x + 12 &= 12x^2 + 9x + 16x + 12 \\ &= (4x + 3)(3x + 4). \end{aligned}$$

Hence the roots of the given quadratic equation are

$$\alpha = -3/4,\ \beta = -4/3.$$

EXAMPLE 2. *Solve the equation*

$$2x^2 + 2\sqrt{6}x + 3 = 0.$$

SOLUTION. We can write

$$\begin{aligned} 2x^2 + 2\sqrt{6}x + 3 &= 2x^2 + \sqrt{6}x + \sqrt{6}x + 3 \\ &= (\sqrt{2}x + \sqrt{3}x)(\sqrt{2}x + \sqrt{3}x). \end{aligned}$$

Thus the roots of the given equation are obtained by solving two identical equations

$$\sqrt{2}x + \sqrt{3} = 0,$$
$$\sqrt{2}x + \sqrt{3} = 0.$$

In such a case we say that the given equation has two identical or coincident roots

$$\alpha = -\sqrt{3}/\sqrt{2},\ \beta = -\sqrt{3}/\sqrt{2}.$$

Example 2 also illustrates an important aspect of the roots of a quadratic equation. The roots of a quadratic equation may be identical. In other words a root of a quadratic equation may repeat itself.

EXERCISE 5.2

Find the zeros of each of the following quadratic polynomials by factoring it.

1. $3x^2 - 8x + 5$
2. $x^2 - 3x - 18$
3. $x^2 - 3x - 10$
4. $6x^2 - x - 12$
5. $18x^2 + x - 5$
6. $2\sqrt{3}\,x^2 + 11x + 4\sqrt{3}$
7. $-\sqrt{35}\;x^2 + 12x - \sqrt{35}$
8. $2x^2 - 3\sqrt{3}\,x + 3$
9. $6x^2 + 41x - 7$
10. $-2x^2 + 4x + 16$
11. $8x^2 + 2x - 3$
12. $3x^2 + 11x - 4$

5.3 METHOD OF COMPLETING THE SQUARE

In section 5.2, we have seen how to solve a given quadratic equation, whenever the corresponding quadratic expression can be written as a product of two linear factors. However, given a quadratic equation, it may be difficult to realise the corresponding linear factors. We shall also see that there are quadratic functions having no factors with real coefficients. But by mere inspection of the quadratic function, we may not be able to decide whether the function has factors with real coefficients.

Let us begin with the quadratic equation

$$x^2 + x - 1 = 0. \tag{1}$$

The quadratic function $x^2 + x - 1$ is negative at $x = 0$ and positive at $x = 1$. This means that the graph of $x^2 + x - 1$ must pass from the y-negative half-plane to the y-positive half-plane when x goes from 0 to 1. It must cut the x-axis somewhere between 0 and 1. These observations show that the equation (1) has a root between 0 and 1. Similarly, $x^2 + x - 1$ is negative at $x = -1$ and positive at $x = -2$. Hence the equation (1) also has a root between -2 and -1. The equation (1) has thus two real roots; one between 0 and 1 and the other between -2 and -1. But from the expression $x^2 + x - 1$, we cannot guess its linear factors.

On the other hand, let us also consider the equation

$$x^2 + x + 1 = 0. \tag{2}$$

Apriori, it is difficult to say whether $x^2 + x + 1$ can or cannot be written as a product of two linear factors with real coefficients. However, we observe that $x^2 + x + 1$ is positive for every real number x. In fact, if $x \geq 0$, then

$$x^2 + x + 1 \geq 1 > 0;$$

if $x \leq -1$, then

$$x^2 + x + 1 = x(x + 1) + 1 \geq 1 > 0$$

since x and $x + 1$ are both negative; if $-1 < x < 0$, then

$$x^2 + x + 1 = x^2 + (x + 1) > 0 + 0 = 0.$$

Thus the graph of $x^2 + x + 1$ does not meet the x-axis. In other words the equation has no roots in real numbers. This implies that $x^2 + x + 1$ cannot be written as a product of two linear factors with real coefficients (see also Chapter 10).

There is a general way of deciding whether a quadratic equation

$$ax^2 + bx + c = 0, a \neq 0 \tag{3}$$

has a real root, and a method of finding the roots of the equation (3). This method is known as the *method of completing the square.* The central idea of the method is to convert (3) to an equation of the form

$$(dx + e)^2 + f = 0.$$

where d, e and f are real numbers. This can be accomplished by considering the equation

$$4a^2x^2 + 4abx + 4ac = 0. \tag{4}$$

Since $a \neq 0$, equations (3) and (4) have the same solutions.

But (4) can be written as

$$(2ax + b)^2 + 4ac - b^2 = 0. \tag{5}$$

Thus we can take

$$d = 2a, e = b, f = 4ac - b^2.$$

Hence a number α(real or complex) is a root of (3) iff it is also a root of

$$(2ax + b)^2 = b^2 - 4ac. \tag{6}$$

We can use (6) to find the roots of (3).

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic equation (3), or the discriminant of the quadratic polynomial $ax^2 + bx + c$. The nature of the roots of (3) can be completely determined by its discriminant $b^2 - 4ac$.

Theorem 1. The quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

where a, b and c are real numbers, has real roots if the discriminant D given by

$$D = b^2 - 4ac$$

is non-negative. In the case when $D \geq 0$, the roots of the given quadratic equation are

$$\alpha = \frac{-b + \sqrt{D}}{2a}, \quad \beta = \frac{-b - \sqrt{D}}{2a}$$

Proof. We have seen in the preceeding discussions that the given quadratic equation and the equation

$$(2ax + b)^2 = b^2 - 4ac = D \tag{6}$$

have the same set of solutions. Suppose the given equation has a real root x_0. Then $(2ax_0 + b)^2 \geq 0$ and x_0 is also a root of (6). This means that $D \geq 0$ is necessary for equation (3) to have a real root. Conversely, assume now that $D \geq 0$. Then D has two real square roots, $+\sqrt{D}$ and $-\sqrt{D}$. Hence taking the square root in (6), we get two linear equations

$$2ax + b = \sqrt{D}$$

$$2ax + b = -\sqrt{D} \tag{7}$$

(compare this with our observation in 5.2). Solving these linear equations, we get two roots of the equation (6).

Since the equations (6) and (3) have the same set of roots, α and β are also the roots of the given quadratic equation (3). ❑

Remark. If $D = 0$ then the roots of the quadratic equation (3) are

$$\alpha = -b/2a, \beta = -b/2a$$

which are identical. On the other hand if $D \neq 0$, then $\alpha\ \alpha \neq \beta$; also $\alpha = \beta$ implies $D = 0$. Thus the equation (3) has coincident roots iff its discriminant vanishes.

EXAMPLE 1. *Find the roots of the equation*

$$x^2 + x - l = 0.$$

SOLUTION. The discriminant D of the quadratic polynomial is

$$D = 1^2 - 4 \times 1 \times (-1) = 5$$

Thus $D > 0$ and the given equation has two real distinct roots. They are given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-1+\sqrt{5}}{2},$$

$$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-1-\sqrt{5}}{2}$$

We also observe that $2 < \sqrt{5} < 3$. Hence the estimates

$$\frac{1}{2} < \frac{-1+\sqrt{5}}{2} < 1,$$

$$-2 < \frac{-1-\sqrt{5}}{2} < \frac{-3}{2}$$

are true. This conforms with our earlier observations that the given equation has a root between 0 and 1 and a root between -2 and -1.

EXAMPLE 2. *Solve the equation*

$$12x^2 + 25x + 12 = 0.$$

SOLUTION. Earlier, in example 1 of section 5.2, we have found the roots of this equation by factoring the corresponding quadratic function,

$$12x^2 + 25x + 12 = (4x + 3)(3x + 4)$$

to get the roots $\alpha = -3/4$ and $\beta = -4/3$. The same conclusion can be drawn using the method of this section. Since $a - 12$, $b = 25$ and $c = 12$, the discriminat is

$$D = b^2 - 4ac = 25^2 - 4 \times 12 \times 12 = 49.$$

Thus $D > 0$ and the equation has two real, distinct roots. These are

$$\gamma = \frac{-b+\sqrt{D}}{2a} = \frac{-25+\sqrt{49}}{24}$$

$$= -18/24 = -3/4,$$

and $$\delta = \frac{-b-\sqrt{D}}{2a} = \frac{-25-\sqrt{49}}{24} = \frac{-32}{24} = -4/3.$$

We see that $\alpha = \gamma$ and $\beta = \delta$.

EXAMPLE 3. *Determine whether the equation*

$$4x^2 + 4x + 1 = 0$$

has real roots and solve for them.

SOLUTION. The discriminant of the equation is

$$D = 4^2 - 4 \times 4 \times 1 = 0.$$

Hence we conclude that the given equation has two coincident roots, namely,

$$\alpha = -4/8 = -1/2,$$

$$\beta = -1/2.$$

This can also be inferred from the observation that

$$4x^2 + 4x + 1 = (2x + 1)^2.$$

EXAMPLE 4. *What are the solutions of the equation*

$$x^2 + 2x - 4 = 0$$

SOLUTION. The discriminant is given by

$$D = 2^2 + 16 = 20,$$

so that the given equation has real distinct roots. They are given by

$$\alpha = \frac{-2+\sqrt{20}}{2} = -1 + \sqrt{5}$$

$$\beta = \frac{-2-\sqrt{20}}{2} = -1 - \sqrt{5}.$$

Now we shall take a fresh look at the equation (6), namely

$$(2ax + b)^2 = b^2 - 4ac = D.$$

If $D \geq 0$, we could take square roots on both sides using the fact that every non-negative real number has two real square roots. This would given us two real solution of equation (6) and hence those of equation (3). However, if only we can give a meaning to $\sqrt{D}$ even if $D < 0$, then we will be able to solve the given equation (3) for any real values of a, b and c. If $D < 0$, then we pass to complex numbers and take square roots of D in C. In this case D has two square roots in **C**, viz., $i\sqrt{-D}$ and $-i\sqrt{-D}$. Thus we have the following theorem.

Theorem 2. If the discriminant

$$D = b^2 - 4ac$$

of the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

is negative, then it has two complex roots which are *conjugate* to each other.

Proof. We have seen that the set of solutions of the given quadratic equation is identical with the set of solutions of

$$(2ax + b)^2 = D. \qquad (*)$$

If $D < 0$, then it has two complex square roots $i\sqrt{-D}$ and $-i\sqrt{-D}$. Hence the two roots of (*) and hence of the given quadratic equation are

$$\alpha = \frac{-b+i\sqrt{-D}}{2a},$$

$$\beta = \frac{-b-i\sqrt{-D}}{2a}.$$

Since b and a are real, we see that

$$\beta = \bar{\alpha} \text{ the complex conjugate of } \alpha.$$

Thus the non-real roots of a quadratic equation with real coefficients always occur in pairs, one being the complex conjugate of the other. ❑

Theorems 1 and 2 completely resolve the question of existence of roots of a quadratic equation with real coefficients. If the discriminant $D > 0$, then the given quadratic equation has two distinct real roots. If $D = 0$, then it has two coincident roots. In the case $D < 0$, the given equation has two complex roots, one being the complex conjugate of the other. Combining all these, we can make the following statement.

Any quadratic equation

$$ax^2 + bx + c = 0$$

where a, b and c are real numbers, has exactly two roots. If α is a non-real root of the equation, then $\bar{\alpha}$ is the other root.

EXAMPLE 5. *Solve the equation*

$$3x^2 + 2x + 1 = 0.$$

SOLUTION. The discriminant is

$$D = -8 < 0.$$

Hence the equation has no real roots. The complex roots are given by

$$\alpha = \frac{-b + i\sqrt{-D}}{2a} = \frac{-1 - i\sqrt{2}}{3},$$

$$\beta = \frac{-b + i\sqrt{-D}}{2a} = \frac{-1 + i\sqrt{2}}{3}.$$

EXAMPLE 6. *Find the roots of the equation*

$$3x^2 + 2\sqrt{3}\,x + 2 = 0.$$

SOLUTION. We have $D = (2\sqrt{3})^2 - 4 \times 3 \times 2 = -12 < 0.$

Hence the roots are

$$\alpha = \frac{-1 + i}{\sqrt{3}}, \quad \beta = \frac{-1 - i}{\sqrt{3}}.$$

EXERCISE 5.3

1. Find the roots of the following equations:

   (*a*) $2x^2 + x + 1 = 0$
   (*b*) $9x^2 + 2x - 3 = 0$
   (*c*) $x^2 + 2x + 4 = 0$
   (*d*) $4x^2 + 2x - 1 = 0$
   (*e*) $x^2 + 6x + 6 = 0$
   (*f*) $2x^2 + 5x + 4 = 0$
   (*g*) $4x^2 + 9x + 2 = 0$
   (*h*) $2\sqrt{3}\,x^2 + 4x - \sqrt{3} = 0$
   (*i*) $3x^2 + 9x - 5 = 0$
   (*j*) $x^2 + 5x - 6 = 0.$

2. For each the following equations find the set of values of a for which the equation has real roots:

   (*a*) $ax^2 + 9x - 1 = 0$
   (*b*) $2x^2 + ax + 2 = 0$
   (*c*) $2x^2 + 4x + a = 0$
   (*d*) $2x^2 + ax - a = 0$
   (*e*) $x^2 + (a + 2)x + a = 0$
   (*f*) $ax^2 + 4x + a = 0.$

3. A person goes to a vegetable market to buy vegetables. He finds that the price of carrot is the square of the price of beet-root. He buys 2 kilos of carrot and 3 kilos of beet-root, and tenders 27 rupees. What is the price of a kilo of carrot?
4. Solve for x in the equation
$$2m(1 + x^2) - (1 + m^2)(x + m) = 0.$$
5. Find the values of a for which the equation
$$a^2x^2 + 2(a + 1)x + 4 = 0$$
has coincident roots.
6. Suppose p and q are real numbers which do not take simultaneously the values $p = 0$, $q = 1$. Suppose the equation
$$(1 - q + p^2/2)\,x^2 + p(1 + q)x + q(q - 1) + p^2/2 = 0$$
has two equal roots. Prove that $p^2 = 4q$.

5.4 RELATIONS BETWEEN ROOTS AND COEFFICIENTS

There are very useful relations between the roots of a quadratic equation
$$ax^2 + bx + c = 0 \tag{1}$$
and its coefficients a, b and c. These relations are true irrespective of whether (1) has only real roots or it has complex roots. If the discriminant $D \geq 0$, then the equation (1) has two real roots given by
$$\alpha = \frac{-b + \sqrt{D}}{2a}, \quad \beta = \frac{-b - \sqrt{D}}{2a}. \tag{2}$$
Now we observe that
$$\alpha + \beta = \frac{(-b + \sqrt{D})}{2a} + \frac{(-b - \sqrt{D})}{2a}$$
$$= -2b/2a = -b/a.$$
Similarly,
$$\alpha\beta = \frac{(-b + \sqrt{D})}{2a}\,\frac{(-b - \sqrt{D})}{2a}$$
$$= \frac{b^2 - D}{4a^2} = \frac{4ac}{4a^2} = c/a.$$
In the case $D < 0$, the solutions of (1) are
$$\alpha = \frac{-b + i\sqrt{-D}}{2a}, \quad \beta = \frac{-b - i\sqrt{-D}}{2a} \tag{3}$$
Now an easy computation gives again
$$\alpha + \beta = -b/a, \ \alpha\beta = c/a.$$
Thus we have the following theorem.

Theorem 3. If α and β are the roots of the quadratic equation
$$ax^2 + bx + c = 0$$
then,
$$\alpha + \beta = -b/a,$$
$$\alpha\beta = c/a. \tag{4}$$

EXAMPLE 1. *Find the sum and product of the roots of the equation*
$$5x^2 + 5x + 1 = 0.$$

SOLUTION. If α and β are the roots of this equation, then (4) shows that

$$\alpha + \beta = -5/5 = -1,$$

$$\alpha\beta = 1/5.$$

EXAMPLE 2. *If* α *and* β *are the roots of*

$$x^2 + 4x + 6 = 0,$$

find the values of

(*i*) $1/\alpha + 1/\beta$, (*ii*) $\alpha^3 + \beta^3$, (*iii*) $\alpha/\beta + \beta/\alpha$.

SOLUTION. We have $\alpha + \beta = -4$, $\alpha\beta = 6$.

Hence $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-4}{6} = \frac{-2}{3}.$$

We can write $$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\therefore \quad \alpha^3 + \beta^3 = (-4)^3 - (3 \times 6 \times (-4))$$
$$= -64 + 72$$
$$= 8.$$

And lastly, $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$
$$= \frac{(\alpha + \beta) - 2\alpha\beta}{\alpha\beta}$$
$$= \frac{(-4)^2 - (2 \times 6)}{6} = 4/6 = 2/3.$$

Suppose α and β are the roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

Then, we have $$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$
$$= x^2 + \frac{b}{a}x + a = \frac{1}{a}(ax^2 + bx + c).$$

This leads to the factorization,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \tag{5}$$

Thus, if we know the roots of a quadratic equation, then we can write down a factorization of the corresponding quadratic function.

Conversely, if α and β are two numbers (real or complex), then the most general quadratic equation having α and β as its roots is given by

$$a(x - \alpha)(x - \beta) = 0,$$

where a is a non-zero, real or complex number: If we restrict the coefficient of x^2 to be 1 then we get a unique quadratic equation

$$(x - \alpha)(x - \beta) = 0.$$

This is the same as

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

A quadratic polynomial in which the coefficient of x^2 is 1 is called a *monic quadratic polynomial* (or simply a monic quadratic).

EXAMPLE 3. *If α and β are the roots of the equation*

$$x^2 + \sqrt{2}\,x + 3 = 0,$$

find the monic quadratic having zeros

$$\alpha^2 + \beta^2 \text{ and } 2\,\alpha\,\beta.$$

SOLUTION. The monic quadratic having $\alpha^2 + \beta^2$ and $2\alpha\beta$ as its roots is given by

$$x^2 - (\alpha^2 + \beta^2 + 2\alpha\beta)x + 2\alpha\beta(\alpha^2 + \beta^2). \qquad (*)$$

However, we know that

$$\alpha + \beta = -\sqrt{2}\,,\ \alpha\beta = 3.$$

Hence, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 2 - 6 = -4$

This gives, $\alpha^2 + \beta^2 + 2\alpha\beta = 2$

and $2\alpha\beta(\alpha^2 + \beta^2) = -24.$

Hence, from (*) the required quadratic is

$$x^2 - 2x - 24.$$

EXAMPLE 4. *Find the values of p for which the sum of the squares of the roots of*

$$x^2 + px - 2 = 0$$

is equal to 5.

SOLUTION. If α and β are the roots of the given equation, then

$$\alpha + \beta = -p,\ \alpha\beta = -2.$$

We have to find the values of p for which

$$\alpha^2 + \beta^2 = 5.$$

This gives $5 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + 4.$

$\therefore$ $p^2 = 1.$

Solving for p, we get $p = \pm 1.$

EXAMPLE 5. *Find the values of a for which one of the roots of*

$$x^2 + (2a + 1)x + (a^2 + 2) = 0$$

is twice the other root. Find also the roots of this equation for these values of a.

SOLUTION. We may assume that the roots of the given equation are α and 2α. Using the relations between the roots of a quadratic equation and its coefficients, we have

$$\alpha + 2\alpha = -(2a + 1)$$

and $2\alpha^2 = a^2 + 2.$

The first relation gives

$$\alpha = \frac{-(2a + 1)}{3}.$$

Substituting this value of a in the second relation, we get

$$2(2a + 1)^2 = 9(a^2 + 2).$$

This is the same as

$$a^2 - 8a + 16 = 0.$$

Thus we get a quadratic equation for a and this equation has coincident roots $a = 4$. Thus there is a unique value of a for which the conditions of the problem are fulfilled. Corresponding to this value of a, the given equation reduces to

$$x^2 + 9x + 18 = 0.$$

Now $\quad x^2 + 9x + 18 = (x + 6)(x + 3)$

so that the roots are $\quad \alpha = -6, \beta = -3.$

EXERCISE 5.4

1. Find the sum and product of roots of each of the following quadratic equations

(*a*) $x^2 + 9x - 8 = 0$
(*b*) $\sqrt{2}\,x^2 - 4x + \sqrt{8} = 0$
(*c*) $3x^2 + 9x + 4 = 0$
(*d*) $4x^2 - 8x + 2 = 0$
(*e*) $6x^2 + 7x - 3 = 0$
(*f*) $28 + 31x - 5x^2 = 0$
(*g*) $x^2 - 6(x + 12) = 0$
(*h*) $x(x + 34) + 289 = 0$
(*i*) $(1/3)x^2 - 4x + 2 = 0$
(*j*) $(0.2)x^2 - (0.3)x + (0.1) = 0$

2. If α and β are the roots, compute $\alpha^3 + \beta^3$, $\alpha/\beta + \beta/\alpha$ and $\alpha^2\beta + \alpha\beta^2$ in each case.

(*a*) $4x^2 + 2x - 9 = 0$
(*b*) $x^2 - 8x + 2 = 0$
(*c*) $2x^2 + 6\sqrt{3}\,x + 3 = 0$
(*d*) $4x^2 + \sqrt{5}\,x + 6 = 0$
(*e*) $9 - 3x - (x^2/4) = 0$
(*f*) $23x - 120 - x^2 = 0$
(*g*) $x^2 + 3x - 2(x + 7) = 0$
(*h*) $8x(1 + x) + 11x - 15 = 0.$

3. Find in each case the monic quadratic having α and β as zeros where α and β are given by

(*a*) $\alpha = \sqrt{2}, \beta = \sqrt{3}$
(*b*) $\alpha = 2\sqrt{2}, \beta = \sqrt{2}$
(*c*) $\alpha = 3, \beta = \sqrt{3}$
(*d*) $\alpha = 2 + \sqrt{2}, \beta = 2 - \sqrt{2}$
(*e*) $\alpha = 0.6, \beta = 1.2$
(*f*) $\alpha = 3 + \sqrt{3}/2, \beta = 3 - \sqrt{3}/2$
(*g*) $\alpha = 2 + 3i, \beta = 2 - 3i$
(*h*) $\alpha = \pi, \beta = e$
(*i*) $\alpha = \sqrt{2} + \sqrt{5}\,i, \beta = \sqrt{2} - \sqrt{5}\,i$
(*j*) $\alpha = \sqrt{2}\,i, \beta = -\sqrt{2}\,i$

4. Suppose the sum of the roots of

$$ax^2 - 6x + c = 0$$

is -3 and their product is 2. Find the values of a and c.

5. If one of the roots of

$$2x^2 + bx + 6 = 0$$

is 3, find the other root. Find also the value of b.

6. Find the monic quadratic with roots α and β, if

$$\alpha\beta = -2, \alpha^2 + \beta^2 = 4(\alpha + \beta).$$

7. Find a necessary and sufficient condition involving only the coefficients in order that one of the roots of

$$ax^2 + bx + c = 0, \quad a \neq 0$$

is the square of the other root.

8. Find the values of x for which the roots g and h of the equation

$$t^2 - 8t + x = 0$$

satisfy the condition that

$$g^2 + h^2 = 4.$$

Find also the roots of the equations corresponding to these values of x.

9. Solve the equation

$$x^2 + px + 10 = 0$$

given that the square of the difference of the roots is 9.

10. Given that α and β are the roots of

$$6x^2 - 5x - 3 = 0$$

find the monic quadratic whose roots are $\alpha - \beta^2$ and $\beta - \alpha^2$.

11. Let α and β be the roots of an equation

$$x^2 + px + q = 0$$

and let γ and δ be the roots of

$$x^2 + Px + Q = 0.$$

Express $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$

in terms of the coefficients p, q, P and Q.

12. Find all the values of a for which the equations

$$x^2 + ax + 1 = 0 \quad \text{and} \quad x^2 + x + a = 0$$

have at least one common root.

5.5 PROBLEMS LEADING TO QUADRATIC EQUATIONS

We shall consider here various examples which ultimately lead to quadratic equations. In the world of applications it is not always that quadratic equations directly appear as the problem to be solved. Ingenuity is required to recognize them below the outer veil.

EXAMPLE 1. *Solve the equation*

$$\sqrt{x} = x - 2.$$

SOLUTION. We begin by setting $y = \sqrt{x}$. Then the given equation reduces to a quadratic equation

$$y^2 - y - 2 = 0.$$

The solutions of this equation are

$$y = 2, y = -1.$$

This gives two values of x,

$$x = 4, x = 1.$$

However, as per our convention $\sqrt{x}$ is the positive square root of x whenever $x \geq 0$. Therefore

$$x - 2 = \sqrt{x} \geq 0.$$

The only value of x satisfying this is $x = 4$.

EXAMPLE 2. *Solve the equation*

$$x^4 - 20x^2 + 64 = 0.$$

SOLUTION. Again the equation is not directly quadratic. If we set $y = x^2$, we get a quadratic equation

$$y^2 - 20y + 64 = 0.$$

The solutions are $y = 16$ and $y = 4$. Hence the solutions of the given equation are precisely those of $x^2 = 16$ and $x^2 = 4$. Therefore the solutions are ± 4 and ± 2.

EXAMPLE 3. *A two digit number is four times the sum of its digits and twice the product of its digits. Find the number.*

SOLUTION. Suppose x is in ten's place and y is in unit's place of the two digit number. Then the given number is $10x + y$. The given conditions imply that

$$10x + y = 4(x + y)$$

$$10x + y = 2xy.$$

Substituting the value of y in terms of x from the first equation into the second equation and thus eliminating y we get,

$$x^2 = 3x.$$

Solving for x, we have two solutions

$$x = 0 \quad \text{and} \quad x = 3.$$

If $x = 0$, then $y = 0$ since the first relation gives $y = 2x$. Hence $10x + y = 0$ and this is not a two digit number. If $x = 3$, then $y = 6$ and $10x + y = 36$. Hence the given number is 36.

EXAMPLE 4. *Solve the equation*

$$x + (12/x) = 8.$$

SOLUTION. We observe that α is a solution of the given equation if α is a solution of the quadratic equation

$$x^2 - 8x + 12 = 0.$$

Since we can write

$$x^2 - 8x + 12 = (x - 6)(x - 2),$$

the solutions of the given equation are 6 and 2.

EXAMPLE 5. *Solve the equation*

$$x^2 + 1/x^2 - 8(x - 1/x) + 14 = 0.$$

SOLUTION. Since $\quad x^2 + 1/x^2 = (x - 1/x)^2 + 2,$

the given equation can be written in the form

$$(x - 1/x)^2 - 8(x - 1/x) + 16 = 0.$$

If we put $\quad y = x - 1/x,$

we get a quadratic equation

$$y^2 - 8y + 16 = 0.$$

It has a repeated root $y = 4$. Thus we get the equation

$$x - 1/x = 4.$$

This is the same as

$$x^2 - 4x - 1 = 0.$$

The discriminant of the equation is $D = 20 > 0$. Hence the equation has the following real roots:

$$\alpha = 2 + \sqrt{5}, \beta = 2 - \sqrt{5}.$$

These are precisely the solutions of the given equation.

EXAMPLE 6. *A right angled triangle is such that its hypotenuse is 1 cm. longer than its base and the altitude is 1 cm. shorter than half the base. Find the base, altitude and hypotenuse.*

SOLUTION. Let us denote the lengths of the base, the altitude and the hypotenuse by x, y and h respectively. The given conditions imply that

$$y = (x/2) - 1 \text{ and } h = x + 1.$$

Using Pythagoras's theorem, we get a relation between x, y and h;

$$h^2 = x^2 + y^2.$$

This reduces to $(x + 1)^2 = x^2 + ((x/2) - 1)^2$.

Simplification gives $x^2 - 12x = 0$.

Thus $x - 0$ or $x = 12$. We can reject $x = 0$, since the base of a triangle cannot be '0'. Hence $x = 12$ cm. and this leads to $h = 13$ cm., $y = 5$ cm.

EXERCISE 5.5

1. Solve the following equations

   (a) $x^4 - 10x^2 + 9 = 0$ (b) $x^4 - 6x^2 + 1 = 0$

   (c) $(1 - x)(x + 2)(x + 3) = 9x^2 - x^3 + 4(2 - 7x)$

   (d) $x^3 - x^2 + x - 1 = 0$ (e) $x - 7/x = 6$

   (f) $x + 2/x = 2\sqrt{2}$ (g) $x^2 - 8/x^2 = 7$

   (h) $\sqrt{x} + \sqrt{1+2x} = 1$ (i) $\sqrt{4-x} + \sqrt{x+9} = 5$

   (j) $\sqrt{3x^2+10} + \sqrt{6-x^2} = 6$ (k) $(x^2 + 1/x^2) - 4(x + 1/x) + 6 = 0$

   (l) $(x^2 + 1/x^2) - 8(x - 1/x) + 13 = 0$.

2. The number of square centimetres in the area of a rectangle is the same as the number of centimetres in the perimeter. If the diagonal is $3\sqrt{5}$ cm., find its sides.
3. The sum of two numbers is 6 and the sum of their reciprocals is 3/4. Find the numbers.
4. The sum of an integer and its reciprocal is 10/3. What is the integer?
5. The sum of squares of two numbers is equal to 5 times their sum. The sum of the reciprocals of the squares of these two numbers is equal to 5/18 times the sum of the reciprocals of the given numbers. Find these numbers.
6. The product of two consecutive even integers is equal to 24. Find these integers.
7. Suppose the sides of a right angled triangle are x, $x + 7$ and $x + 8$. Find the area of the triangle.
8. Solve the equation
$$2/x + 5/(x + 2) = 9/(x + 4).$$
9. A farmer has a rectangular garden of total area 80 sq. meters. He requires 36 meters of barbed wire for fencing it. Find the dimensions of the garden.
10. Determine the values of k for which the equation
$$\frac{x^2 + x + 2}{3x + 1} = k$$
has both roots real.
11. Find all integers a such that
$$(x - a)(x - 12) + 2$$
can be factored as $(x + b)(x + c)$ where b and c are integers.

5.6 BEHAVIOUR OF QUADRATIC FUNCTIONS

A function of the form
$$f(x) = ax + b,\ a \neq 0 \tag{1}$$
is called a *linear function*. As we shall observe in chapter 7, the graph of such an expression always represents a straight line on a coordinate plane. As such, its behaviour is completely determined. If this line meets the x-axis at some pointy x_0, and if $f(x_1) < 0$ for some $x_1 < x_0$, then we have

$$ax_0 + b = 0$$
$$ax_1 + b < 0$$

and these imply $\quad ax_0 > ax_1.$

But since $x_1 < x_0$, we must have $a > 0$. This in turn implies

$$f(x) = ax + b < ay + b = f(y)$$

for any $x < y$. Thus f is an increasing function (see Fig. 5.6).

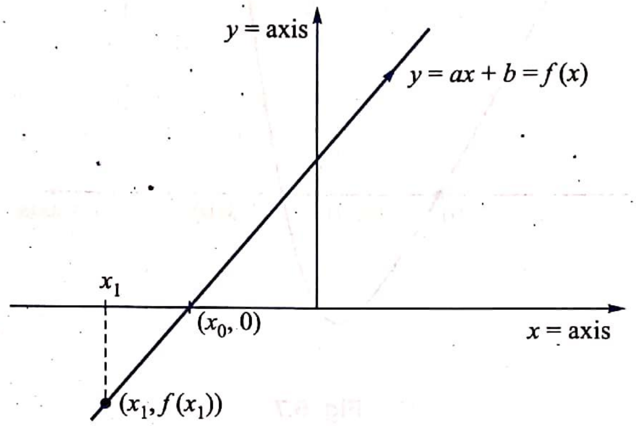


Fig. 5.6

Similarly, if $f(x_1) > 0$ for some $x_1 < x_0$ then f is a decreasing function. The student is advised to carry out the argument.

A function of the form

$$f(x) \equiv ax^2 + bx + c, \quad a \neq 0 \tag{2}$$

is called a *quadratic expression* or a quadratic polynomial. Let us consider the polynomial.

$$3x^2 + 2x + 1.$$

The discriminant of this polynomial is $D = -8$. Hence this polynomial has no real zeros. So the graph of the function

$$f(x) = 3x^2 + 2x + 1$$

does not meet the x-axis. This implies that the graph of $f(x)$ lies completely in the upper half-plane or lies completely in the lower half-plane. In turn, we conclude that $f(x)$ is positive for all values of x or negative for all values of x. But $f(0) = 1 > 0$. Therefore, $f(x) > 0$ for all values of x.

On the other hand, let us consider the polynomial

$$3x^2 + 2x - 1.$$

Its discriminant is 16 so that it has real zeros. These are

$$\alpha = -1, \beta = 1/3$$

Thus the graph of

$$g(x) = 3x^2 + 2x - 1$$

cuts the x-axis at -1 and $1/3$. Since $g(x)$ has no other zeros, the graph of $g(x)$ does not meet the x-axis at any other point (see Fig. 5.7).

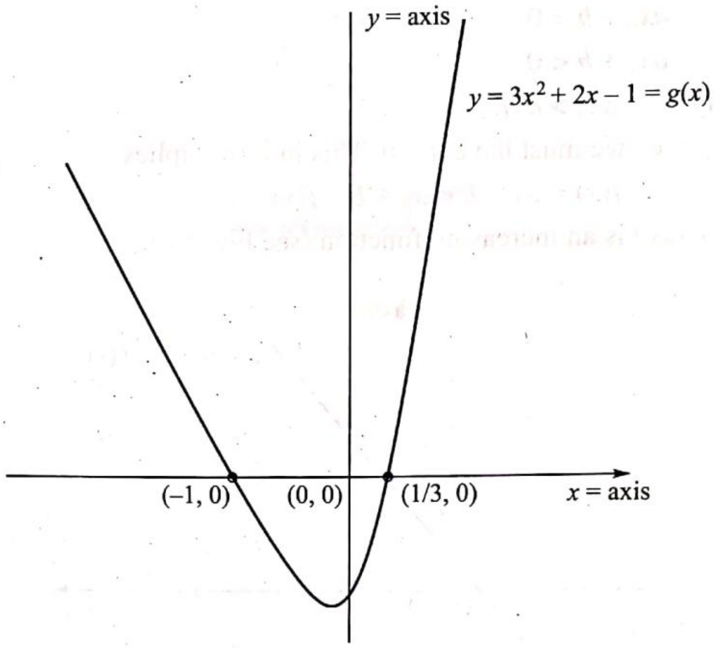


Fig. 5.7

Thus the graph of $g(x)$ passes from one y half-plane to the other y half-plane at both α and β. Since $g(x)$ has no zeros between α and β, the graph of $g(x)$ remains in the same y half-plane for $\alpha < x < \beta$. The factorization

$$3x^2 + 2x - 1 = 3(x - 1/3)(x + 1)$$

shows that

$$g(x) < 0 \quad \text{if } -1 < x < 1/3,$$
$$g(x) > 0 \quad \text{if } x < -1 \text{ or } x > 1/3.$$

Thus the graph of $g(x)$ lies in the lower half-plane for $-1 < x < 1/3$ and it lies in the upper half-plane if either $x > 1/3$ or $x < -1$. We infer that the graph of $g(x)$ passes from positive to negative at α and from negative to positive at β, as we move from left to right along the x-axis.

This type of behaviour is true of any quadratic polynomial. Let us, begin with the general quadratic polynomial

$$f(x) = ax^2 + bx + c, a \neq 0. \tag{3}$$

Let α and β be the roots of the equation

$$ax^2 + bx + c = 0. \tag{4}$$

Suppose the discriminant D given by

$$D = b^2 - 4ac$$

is non negative. Then the equation (4) has real roots; α and β are real in this case. Assume $\alpha < \beta$. Then the equation (4) has no roots either between α and β, or before α, or after β. Hence the graph of $f(x)$ cuts the x-axis at α and β, and at no other point. This implies that the graph of $f(x)$ lies in one of the upper and lower half-planes for $x < \alpha$ and $x > \beta$, and lies in the opposite half-plane for $\alpha < x < \beta$.

Now we can write

$$f(x) = a(x - \alpha)(x - \beta) \tag{5}$$

and therefore

$$af(x) > 0 \quad \text{if either } x < \alpha \text{ or } x > \beta,$$

$$af(x) < 0 \quad \text{if } \alpha < x < \beta.$$

This determines the sign of $f(x)$ provided we know the sign of a. If $a > 0$, then $f(x) > 0$ if $x < \alpha$ or if $x > \beta$ and $f(x) < 0$ if $\alpha < x < \beta$. Thus if $a > 0$, then the graph of $f(x)$ passes from the upper half-plane to the lower half-plane at α and passes from the lower half-plane to the upper half-plane at β. Similarly, in the case $a < 0$, the behaviour of the graph of $f(x)$ is reversed. Thus the sign of a completely determines the behaviour of the quadratic expression (3), provided the equation (4) has only real roots.

Suppose $D < 0$. Then the equation (4) has two roots α and $\overline{\alpha}$, where $\overline{\alpha}$ is; the complex conjugate of α. If α is of the form

$$\alpha = s + it$$

where s and t are real numbers, then

$$\overline{\alpha} = s - it.$$

We can write again

$$\begin{aligned} f(x) &= a(x - \alpha)(x - \overline{\alpha}) \\ &= a(x - s - it)(x - s + it) \\ &= a\{(x - s)^2 + t^2\}. \end{aligned}$$

The expression in the braces is positive for every value of x. Hence $f(x) > 0$ for all x if $a > 0$, and $f(x) < 0$ for all x if $a < 0$. This shows that the sign of $f(x)$ is completely determined by the sign of a. We record these observations.

Any quadratic expression $f(x) \equiv ax^2 + bx + c$ has the same sign between its real zeros and changes sign only when the graph of $f(x)$ passes through any of its real zeros. If $f(x)$ has no real zeros, then $f(x)$ has the same sign for all real values and the sign of $f(x)$ is determined by the sign of the coefficient a.

If the equation

$$ax^2 + bx + c = 0$$

has real roots α and β, then we have the factorization

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \tag{6}$$

Thus the quadratic polynomial $ax^2 + bx + c$ can be written as a product of linear factors, each linear factor having only real coefficients. If the given equation has no real roots, then a factorization of the form (8) with α and β real is impossible. Nevertheless, we can find a factorization of the form (8) with complex α and β (In fact $\beta = \overline{\alpha}$). If the quadratic expression $ax^2 + bx + c$ is such that it has no factorization of the form (8) with real α and β, then we say the polynomial $ax^2 + bx + c$ is, *irreducible* over **R**. Since the given equation has real roots if and only if the discriminant is non-negative, we conclude that *$ax^2 + bx + c$ is irreducible over* **R** *if and only if the discriminant $D = b^2 - 4ac$ is negative.*

All the relevant properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ are given in Table 5.1.

TABLE 5.1

Sl. NO.	a	D	roots of $f(x) = 0$	Sign of $f(x)$
1.	$a > 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) > 0$ for $x < \alpha$ < 0 for $\alpha < x < \beta$ > 0 for $x > \beta$
2.	$a > 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) > 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ > 0 for $x > \alpha$
3.	$a > 0$	$D < 0$	Complex roots $\alpha, \bar{\alpha}$	$f(x) > 0$ for all x
4.	$a < 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) < 0$ for $x < \alpha$ > 0 for $\alpha < x < \beta$ < 0 for $x > \beta$
5.	$a < 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) < 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ < 0 for $x > \alpha$
6.	$a < 0$	$D < 0$	Complex roots $\alpha, \bar{\alpha}$	$f(x) < 0$ for all x

EXAMPLE 1. *Find the values of x for which the inequality*

$$x^2 - x - 2 < 0$$

is true.

SOLUTION. We observe that $x^2 - x - 2 = (x - 2)(x + 1)$.

Hence the inequality is true if the values of the two factors have opposite signs. Thus either $x - 2 < 0, x + 1 > 0$ or $x - 2 > 0, x + 1 < 0$. There is no x for which $x - 2 < 0$, $x + 1 > 0$ is true. Hence the set of values of x for which the given inequality is true is $-1 < x < 2$. We can also infer this from Table 1. The leading coefficient $a > 0$ and the discriminant $D > 0$. The zeros of $x^2 - x - 2$ are -1 and 2. Hence $x^2 - x - 2 < 0$ only between the zeros -1 and 2; *i.e.*, $-1 < x < 2$.

EXAMPLE 2. *Find all the values of x for which the inequality*

$$\frac{x^2 - 2x - 1}{x + 1} < x$$

holds.

SOLUTION. The given inequality is equivalent to

$$\frac{x^2 - 2x - 1}{x + 1} - x < 0.$$

Hence it is sufficient to find those x for which

$$\frac{-3x - 1}{x + 1} < 0$$

is true. Equivalently, it is sufficient (Why ?) to consider the inequality

$$-(3x + 1)(x + 1) < 0.$$

Hence either both $(3x + 1)$ and $(x + 1)$ must be positive or both must be negative. Hence the set of values of x for which the given inequality holds is $x > -1/3$ and $x < -1$.

EXERCISE 5.6

Determine in each of the following problems the set of values of x for which the given inequality is true.

1. $x^2 + 4x + 12 > 0$
2. $x^2 - 4x + 3 < 0$
3. $9x^2 - 6x + 1 \geq 0$
4. $(1/4)x^2 - 3(x + 5) < 0$
5. $x(x - 3) - 2 < 3x - (x^2 + 2)$
6. $x(x - 1) - 6 > 5x - x^2$
7. $21 - 7(2x - 9) > 7x^2$
8. $1/3\ (x - 2) < 3/5(x^2 + 4/3)$
9. $3x^2 - 5x - 8 \leq 0$
10. $(x^2 - 16x)^2 - 63 \geq 2(x^2 - 16x)$
11. $(x^2 - 4)(x^2 - x - 2) \leq 0$
12. $\dfrac{(x-1)(x-7)}{x+4} > 0$
13. $\dfrac{2x-1}{(x+4)(x-3)(5-x)} \geq 0$
14. $\dfrac{x^2-7x+12}{2x^2+4x+5} > 0$
15. $2 + \dfrac{7x-5}{8x+3} \leq 0$
16. $\dfrac{x^2-9}{3x-x^2-24} < 0$
17. $\dfrac{x-1}{x} - \dfrac{x+1}{x-1} < 2$
18. $\dfrac{1}{x-2} + \dfrac{1}{x-1} > \dfrac{1}{x}$
19. $\dfrac{3}{x+1} + \dfrac{7}{x+2} \leq \dfrac{6}{x-1}$
20. $|x^2 + 3x| \geq 2 - x^2$
21. $\dfrac{4}{|x+3|-1} \geq |x+2|$

PROBLEMS

1. Let $p(x)$ and $q(x)$ be two quadratic polynomials with integer coefficients. Suppose they have a non-rational zero in common. Show that

$$p(x) = rq(x)$$

for some rational number r.

2. Let a, b, c be integers, and suppose the equation

$$f(x) = ax^2 + bx + c = 0$$

has an irrational root r. Let $u = p/q$ be any rational number such that $|u - r| < 1$. Prove that

$$\left(\frac{1}{q^2} \leq |f(u)| \leq K|u - r|\right)$$

for some constant K. Deduce that there is a constant M such that

$$\left| r - \frac{p}{q} \right| \geq M/q^2$$

(This is useful in approximating the non-rational zero of a polynomial by a rational number)

3. Find the value of the positive integer n for which the quadratic equation

$$\sum_{k=1}^{n} (x+k-1)(x+k) = 10n$$

has solutions α and $\alpha + 1$ for some α.

4. Let $p(x)$ be a monic quadratic polynomial over **Z**. Show that, for any integer n, there exists an integer k such that

$$p(n)p(n+1) = p(k).$$

5. If the equations

$$x^2 + bx + c = 0, \quad bx^2 + cx + 1 = 0$$

have a common root, prove that either

$$b + c + 1 = 0$$

(or) $\quad b^2 + c^2 + 1 = bc + b + c.$

6. If the roots of the equation $x^2 + bx + c = 0$ are real, show that the roots of the equation

$$x^2 + bx + c(x + a)(2x + b) = 0$$

are again real for every real number a.

7. Find all positive integers n for which the quadratic equation

$$a_{n+1}x^2 - 2x\sqrt{\sum_{k=1}^{n+1} a_k^2} + \sum_{k=1}^{n} a_k = 0$$

has real roots for every choice of real numbers $a_1, a_2, \ldots, a_{n+1}$.

8. Let the polynomial $p(z) = z^2 + az + b$ be such that a and b are complex numbers and $|p(z)| = 1$ whenever $|z| = 1$. Prove that $a = 0$ and $b = 0$.

9. Find necessary and sufficient conditions on the coefficients a, b, w so that the roots of the equations

$$z^2 + 2az + b = 0, \quad z - w = 0$$

are collinear in the plane.

10. Find necessary and sufficient conditions an a, b, c, d so that the equations.

$$z^2 + az + b = 0, \quad z^2 + cz + d = 0$$

are collinear in the plane.

11. Determine all quadratic polynomials $p(x)$ with complex coefficients such that the $\alpha + i\beta$ is a zero of $p(x)$ iff $-\alpha + i\beta$ is a zero of $p(x)$ (Here α and β are real numbers).

12. Let $ax^2 + bx + c$ be a quadratic polynomial with real coefficients such that

$$|ax^2 + bx + c| \le 1 \quad \text{for} \quad 0 \le x \le 1,$$

Prove that

$$|a| + |b| + |c| \le 17. \qquad (*)$$

Show that the equality can hold in (*) by constructing an example.

13. Let $p(x)$ be a quadratic polynomial such that for distinct reals α and β,

$$p(\alpha) = \alpha, \; p(\beta) = \beta$$

show that α and β are roots of

$$p[p(x)] - x = 0$$

and find the remaining roots.

14. Solve the equation $\sqrt{a - \sqrt{a + x}} = x$

15. Find the real roots of

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$$

16. Let $p(x)$ be a quadratic polynomial

$$p(x) = ax^2 + bx + c$$

such that $|p(x)| \leq 1$ for $|x| \leq 1$. Prove that

$$|cx^2 + bx + a| \leq 2$$

for $|x| \leq 1$.

17. Suppose α is a real root of the equation

$$ax^2 + bx + c = 0$$

and β is a real root of the equation

$$-ax^2 + bx + c = 0$$

show that the equation

$$(a/2)x^2 + bx + c = 0$$

has a root lying between α and β.

18. Let $p(x) = ax^2 + bx + c$ be such that $p(x)$ takes real values for real values of x and nonreal values for nonreal values of x. Prove that $a = 0$.

19. Consider the number

$$\alpha = \left(\frac{n+\sqrt{n^2-4}}{2}\right)^m$$

where $n \geq 2$ and m are natural numbers. Prove that

$$\alpha = \frac{k+\sqrt{k^2-4}}{2}$$

for some natural number k.

20. Let $p(x) = ax^2 + bx + c$ be a polynomial in $\mathbf{R}(x)$ such that $|p(\alpha)| \leq 1$ for $|\alpha| \leq 1$. Prove that $|2a\alpha + b| \leq 4$ for $|\alpha| < 1$.

21. If a and b are positive reals, prove that

$$\frac{1}{x}+\frac{1}{x-a}+\frac{1}{x-b} = 0$$

has two real roots, one between $a/3$ and $2a/3$, and another between $-2b/3$ and $-b/3$

22. If a, b, c, p, q, r are real numbers such that

$$ax^2 + bx + c \geq 0,$$
$$px^2 + qx + r \geq 0$$

for all real numbers prove that

$$apx^2 + bqx + cr \geq 0$$

for all real x.

CHAPTER

6

TRIGONOMETRY

6.1 INTRODUCTION

Trigonometry means 'measurement of Triangles'. It is a word derived from Gonia, a Greek word, meaning, an angle. This science was nurtured on Indian soil for a number of centuries by Hindu scholars like Aryabhata (5th Cen. A.D.), Varahamihira (6th Cen. A.D.), Brahmagupta (7th Cen. A.D.) and Bhaskara (12th Cen. A.D.); but later it passed on to the west through the Arabs. After the 16th Century modern Trigonometry took shape out of the works of European mathematicians like Vieta (16th Cen. A.D.), Euler (18th Cen. A.D.) and others. Though originally it was confined to a study of the relations between the sides and angles of a triangle, in modern times, it has given rise to mathematical functions of angular magnitudes, through the medium of which many kinds of geometrical and algebraic investigations are carried out in every branch of Mathematics and its application. We shall begin the study of Trigonometry by a comment on one of the most fascinating numbers in Mathematics, namely, the irrational number, π.

Take a circle of any size. Measure, if you can, the length l of the circumference of the circle. Measure also the diameter d of the circle. The ratio l/d is always the same whatever be the circle. This is a mathematical fact, but the proof of this is beyond the scope of this book. This constant number l/d or

$$\frac{\text{Circumference of a circle}}{\text{Diameter of the same circle}}$$

is denoted by π. Its approximate values are

$$\frac{22}{7}, \frac{355}{113}, 3.1416, \textit{etc.}$$

Consider now a circle, centre O, radius r. Let A be a fixed point on the circle and P a variable point. Suppose P is initially at A and moves along the circle, say in the anticlockwise direction. By the symmetry of the circle, we may assume that the angle AOP subtended by arc AP at O is proportinal to the length of the arc itself (we cannot assume such a thing, for example, for an ellipse, in which equal arcs generally subtend unequal angles at the centre).

Definition. In a circle, the angle subtended at the centre by an arc of length equal to the radius of the circle is called a *radian.*

Consider a circle centre O and radius r (see Figure 6.1). The circumference of the circle itself can be considered as an arc of the circle and since its length is $\pi \times$ diameter, that

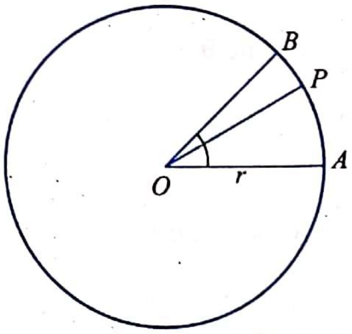


Fig. 6.1

is, $2\pi\, r$, it subtends an angle of 2π radians at the centre. But this angle is also equal to 360 degrees. Hence

$$2\,\pi \text{ radians} = 360 \text{ degrees},$$

or $$\pi \text{ radians} = 180 \text{ decrees}.$$

Notation x radians is denoted by the symbol x^c.

Thus $1^c = \dfrac{180°}{\pi}$ and $1° = \dfrac{\pi^c}{180}$. Here 'c' stands for circular measure:

From the first of these relations, it follows that one radian is $57°\ 17'\ 45''$ approximately. [1 degree = 60 minutes = $60'$ and 1 minute = 60 seconds = $60''$].

Recall the representation of angles in the xy-plane, of various sizes, and of both positive and negative directions, described in the beginning of Chapter 3. In particular, also note that if two or more angles have the same final position in such a representation (*e.g.*, 40°, 400°, and –320°), they are said to be *coterminal.*

CONVENTION : In connection with angles, if no unit of measurement is mentioned, the radian is to be understood,

EXERCISE 6.1

1. Let in a circle, radius r, centre O, an arc AB of length l subtend an angle θ^c at the centre. Show that (a) $l = r\theta$ and (b) area of sector $OAB = (1/2)\, r^2\theta$. What are the corresponding formulae if angle $AOB = \theta°$?
2. Two cities lying on the equator are separated by a distance of 120 miles. What is the longitudinal difference between them. (Radius of the equator may be taken as 4000 miles.)?
3. What is the angle in radians between the hands of a clock when the time is (a) 3.20 (b) 4.20?
4. Assume that the moon's radius is 1800 km. If it subtends an angle of $32'$ at the eye, what is its distance from the observer? What assumptions are you making?

6.2 TRIGONOMETRIC FUNCTIONS OR RATIOS

Let θ be any angle positive, negative or zero represented in the xy-plane by angle AOB.

Choose a point P on this final position $\overrightarrow{OB}$, $P \neq O$ and let $P = (x, y)$, $OP = r$. We shall take r to be positive always. The *Trigonometric (or circular) functions* of θ are defined as follows:

$$\text{sine } \theta = \frac{y}{r}, \text{ cosine } \theta = \frac{x}{r},$$

$$\text{tangent } \theta = \frac{y}{x}, \text{ cotangent } \theta = \frac{x}{y},$$

$$\text{secant } \theta = \frac{r}{x}, \text{ cosecant } \theta = \frac{r}{y}.$$

These functions are abbreviated to sin θ, cos θ, tan θ, cot θ, sec θ and cosec θ. When $x = 0$, tan θ and sec θ are undefined. When $y = 0$, cot θ and cosec θ are undefined.

When θ is an acute angle we can define these functions in terms of the sides of a right triangle (figure 6.2):

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{OP} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{OQ}{OP} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{OQ} = \frac{y}{x},$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{OQ}{PQ} = \frac{x}{y},$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{OP}{OQ} = \frac{r}{x},$$

$$\text{cosec }\theta = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{OP}{PQ} = \frac{r}{y}.$$

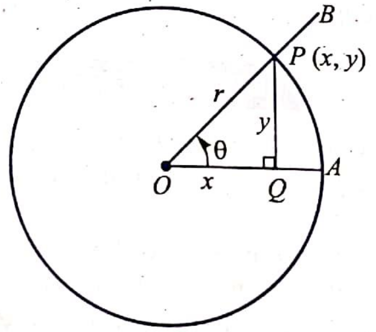


Fig. 6.2

Notation. The powers $(\sin\theta)^n$, $(\cos\theta)^n$ ... are usually written as $\sin^n\theta$, $\cos^n\theta$,...

Basic Identities satisfied by the circular functions:

If θ is any angle,

I. (*a*) $\sin\theta \text{ cosec }\theta = 1$, (*b*) $\cos\theta \sec\theta = 1$,

(*c*) $\tan\theta \cot\theta = 1$, (*d*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,

(*e*) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

II. (*a*) $\sin^2\theta + \cos^2\theta = 1$, (*b*) $1 + \tan^2\theta = \sec^2\theta$,
(*c*) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

These relations make sense only for those values of θ for which the functions involved are defined.

Proof. I (*a*) – (*e*) follow from the definitions.

II follows from the relation $x^2 + y^2 = r^2$, which is obtained from Pythagoras's Theorem. (See Fig. 6.2).

EXERCISE 6.2

1. Show that for any angle θ, $|\sin\theta| \le 1$, $|\cos\theta| \le 1$, $|\sec\theta| \ge 1$, $|\operatorname{cosec}\theta| \ge 1$.
2. Find the set of values of θ for which the following ratios are not defined, (*a*) tan θ (*b*) cot θ (*c*) sec θ (*d*) cosec θ.
3. Show that the trigonometrical ratios of coterminal angles are equal.
4. Find the ratios of 0, 90°, 180°, 270°, 360° and verify that your values agree with the following table:

TABLE 6.1

angles / *ratios*	*0*	*90°*	*180°*	*270°*	*360°*
sine	0	1	0	– 1	0
cosine	1	0	– 1	0	1
tangent	0	not defined	0	not defined	0
cotangent	not defined	0	not defined	0	not defined
secant	1	not defined	– 1	not defined	1
cosecant	not defined	1	not defined	– 1	not defined

Prove the identities (5) – (12)

5. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \,.\, \operatorname{cosec}^2\theta = (\tan\theta + \cot\theta)^2$.
6. $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 5 + (\tan\theta + \cot\theta)^2$.
7. $\dfrac{1-\cos A}{1+\cos A} = (\operatorname{cosec} A - \cot A)^2$.
8. $\dfrac{\tan A - \sec A + 1}{\tan A + \sec A - 1} = \dfrac{1-\sin A}{\cos A}$.
9. $(1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x)$.
10. $\dfrac{\sin^2\theta}{1-\cot\theta} + \dfrac{\cos^2\theta}{1-\tan\theta} = 1 + \sin\theta\cos\theta$.
11. $\dfrac{\sin^4\theta}{1-\cot\theta} + \dfrac{\cos^4\theta}{1-\tan\theta} = 1 + \sin\theta\cos\theta - \sin^2\theta\cos^2\theta$
12. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.
13. Consider an isosceles right triangle *OAB*, in which $\angle AOB = 90°$, $OA = OB = a$. Use Pythanoras's theorem to evaluate *AB* and hence find the rations of 45°. The values are given in Table 6.2.

14. Consider an equilateral triangle ABC in which each side is $2a$. Draw AD perpendicular to BC. Use Pythagoras's theorem to find AD and hence find the ratios of 30° and 60°. The values are given in Table 6.2.

TABLE 6.2

Ratio / *Angle*	*sine*	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
30°	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$

15. Show that the definitions of the ratios are independent of the choice of the point P on $\vec{OB}$ (figure 6.2).

16. Find the signs of the six trigonometric ratios of angles in different quadrants and verify that the signs agree with the following table.

TABLE 6.3

Quadrant	*Sine*	*Cosine*	*Tangent*	*Cotangent*	*Secant*	*Cosecant*
First quadrant	+	+	+	+	+	+
Second quadrant	+	−	−	−	−	+
Third quadrant	−	−	+	+	−	−
Fourth quadrant	−	+	−	−	+	−

17. If $\cos\theta = k$ and θ is an angle in the second quadrant, determine the remaining ratios of θ.

18. If $\tan\theta = \dfrac{-3}{4}$ and $3\pi/2 < \theta < 2\pi$, evaluate $\dfrac{16 - 2\sin\theta + \cos\theta}{13 + 4\sec\theta + 6\,\mathrm{cosec}\,\theta}$.

6.3 TRIGONOMETRICAL RATIOS OF 90° ± θ, 180° ± θ, 270° ± θ, 360° ± θ, – θ

In figures 6.3, 6.4, 6.5 and 6.6, the position of $90° - \theta$ is shown for various positions of θ.

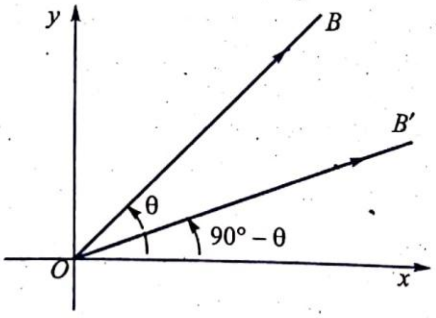


Fig. 6.3

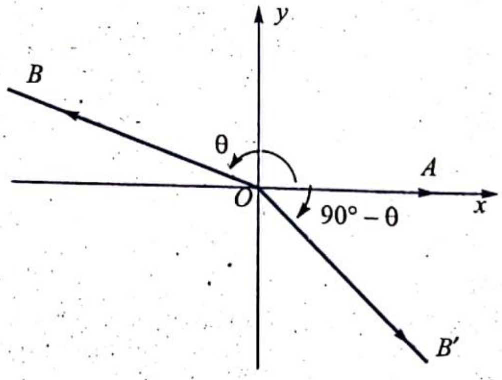


Fig. 6.4

The reader is advised to draw similar figures showing positions of $90° + \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$, $-\theta$ for various positions of θ, *i.e.*, for positions of θ in each of the four quadrants. Now we wish to express the ratios of $90° - \theta$ in terms of those of θ themselves.

Let us take the case when $90° < \theta < 180°$, that is, when θ is in the second quadrant. If angle AOB represents θ, then angle $AOB' = 90° - \theta$ is in the fourth quadrant as shown in Figure 6.7. If P and Q are points on $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ respectively such that $OP = OQ = r$, and P' and Q' are the feet of perpendiculars from P and Q respectively to the y-axis and x-axis, then it is easy to see that $\Delta P'OP$ is congruent to $\Delta Q'OQ$. So $|OP'| = |OQ'|$ and $|PP'| = |QQ'|$. Consequently if $P = (x, y)$, then $Q = (y, x)$. Hence

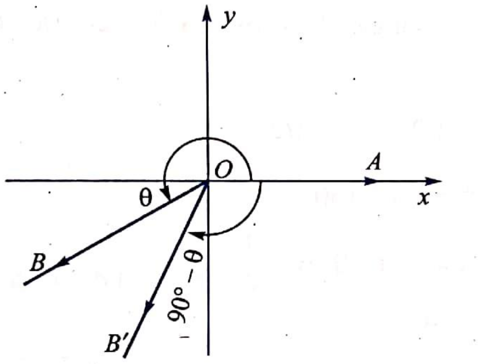


Fig. 6.5

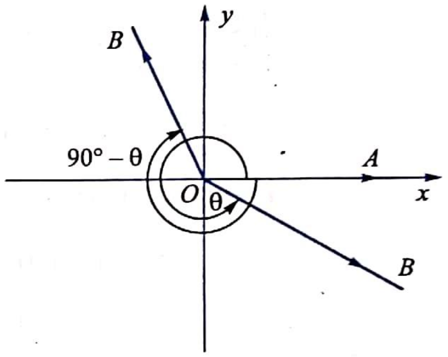


Fig. 6.6

$$\sin(90° - \theta) = x/r = \cos\theta,$$
$$\cos(90° - \theta) = y/r = \sin\theta,$$
$$\tan(90° - \theta) = x/y = \cos\theta,$$
$$\cot(90° - \theta) = y/x = \tan\theta,$$
$$\sec(90° - \theta) = r/y = \operatorname{cosec}\theta,$$
$$\operatorname{cosec}(90° - \theta) = r/x = \sec\theta.$$

Just using the relation $\sin(90° - \theta) = \cos\theta$, one can deduce the other five relations (how?). The same relations can be proved to be true, when θ is in any other quadrant. Similarly the ratios of $90° + \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$, $(-\theta)$ can be obtained in terms of those of θ. Table 4 helps one to read off these ratios.

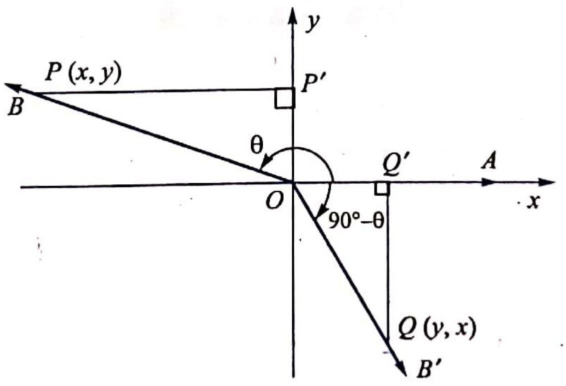


Fig. 6.7

The ratios of $360° + \theta$ and θ are the same since these two angles are co-terminal and so are the ratios of $360° - \theta$ and $-\theta$ for the same reason. The reader is advised to prove the validity of all these relations for at least one position of θ. If we look at the table, we see that ***the ratio itself changes in the presence of 90° and 270°*** and does not in the other cases. Further the sign depends on the angle we are considering.

The table also helps us to reduce the ratios of any angle to those of an acute angle; in fact to those of an angle θ where $0 \le \theta \le 45°$ (how?).

EXAMPLE 1. (*a*) $\sin 135° = \sin (180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}$.

(*b*) $\sec 240° = \sec (270° - 30°) = -\operatorname{cosec} 30° = -2$.

(*c*) $\tan (\theta - 180°) = -\tan (180° - \theta) = \tan \theta$.

(*d*) If A, B, C are the angles of a triangle, then $\cos (A + B) = \cos (\pi - C) = -\cos C$ and

$$\sin\left(\frac{B+C}{2}\right) = \sin (\pi/2 - A/2) = \cos, A/2.$$

(*e*) $\sin (-1230°) = -\sin 1230° = -\sin 150°$.

$$= -\sin (180° - 30°) = -\sin 30° = -\frac{1}{2}.$$

TABLE 6.4

Ratio / Angle	*sine*	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
$90° - \theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$
$90° + \theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$	$-\tan\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$
$180° - \theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$	$-\cot\theta$	$-\sec\theta$	$\operatorname{cosec}\theta$
$180° + \theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$\cot\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$
$270° - \theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$	$\tan\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$
$270° + \theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$	$-\tan\theta$	$\operatorname{cosec}\theta$	$-\sec\theta$
$360° - \theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$
$360° + \theta$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$

EXERCISE 6.3

1. Simplify

(*a*) $$\frac{\sin(180° + \theta)\cos(270° - \theta)\cot(\theta - 360°)}{\cos(\theta - 90°)\sin(360° - \theta)\tan(270° + \theta)}.$$

(*b*) $$\frac{\sin^2(90° - \theta) + \sin^2(90° + \theta) - 1}{1 - \cos^2(270° - \theta) - \cos^2(270° + \theta)}.$$

(*c*) $$\frac{\sin(540° - A)\cos(-90° + A)\tan(270° - A)}{\operatorname{cosec}(1170° + A)\sec(540° + A)\cot(-90° - A)}.$$

2. Express as trigonometric ratios of acute angles and determine their values.

   (a) sin 1410° (b) cos (– 2040°) (c) tan (– 510°).

   (d) sec 56π/3 (e) cot (– 11π/6) (f) cosec 23π/3.

3. If $ABCD$ is a quadrilateral, then show that

   (a) $\sin(A + B) + \sin(C + D) = 0$, (b) $\cos\dfrac{B+C}{2} + \cos\dfrac{A+D}{2} = 0$.

   (c) $\tan\dfrac{A+C}{4} = \cot\dfrac{B+D}{4}$.

4. Show that if n is any integer, then

   (a) $\sin(n\pi + (-1)^n A) = \sin A$. (b) $\cos(2n\pi \pm A) = \cos A$.

   (c) $\tan(n\pi + A) = \tan A$. (d) $\sin n\pi = \tan n\pi = 0$, $\cos n\pi = (-1)^n$.

   (e) $\sin(2n + 1)\pi/2 = (-1)^n$, $\cos(2n + 1)\pi/2 = 0$.

   (f) $\sin(n\pi + A) = (-1)^n \sin A$, $\cos(n\pi + A) = (-1)^n \cos A$.

5. Show that

   $\tan 1° \tan 2° \tan 3° \ldots \tan 89° = 1$.

6.4 FUNCTIONS AND THEIR GRAPHS

Recall that a function f from a set A to a set B, denoted by $f : A \to B$, associates to each element of A a unique element of B. The set A is called the *domain* of f and the set B the *co-domain* of f. If x is an element of A which is associated by f with (or mapped to) the element y of B, we say y is the *image of* x under f and write $f(x) = y$.

Here are a few examples.

EXAMPLE 1. Let $A = \{a, b, c, d, e\}$ and $B = \{p, q, r, s, t, u\}$. Let $f : A \to B$ be given by $f(a) = r, f(b) = p, f(c) = r, f(d) = q, f(e) = u$. We may represent f by a Venn diagram as in Fig. 6.8.

The set of elements in B which are images of elements in A is called the *range of f* and is denoted by Ran f, Im f or $f(A)$. That is, range of $f = \{f(x) \mid x \in A\} \subset B$. In the foregoing example, $Imf = \{p, q, r, u\}$.

EXAMPLE 2. Let $f : \mathbf{N} \to \mathbf{Z}$ be defined by $f(n) = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ +1, & \text{if } n \text{ is even.} \end{cases}$

Here $Imf = \{-1, 1\}$.

We are generally interested in functions defined on **R** or its subsets such as intervals (open, closed or semiopen) or their union and taking values in **R**.

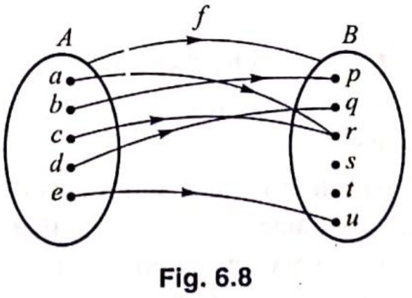


Fig. 6.8

EXAMPLE 3. Let $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$ for x in $\mathbf{R}$. We observe that $Im f = [0, \infty)$.

We say a function $f: A \to B$ is *one-one* iff different elements in A have different images, that is, $x_1, x_2 \in A, x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; or equivalently, $x_1, x_2 \in A, f(x_1) = f(x_2)$ implies $x_1 = x_2$. The function f in Example 3 is not one-one, because, for instance, $f(2) = f(-2) = 4$.

Question: Which of the functions in Examples 1 and 2 are one-one?

We say that a function $f: A \to B$ is *onto* iff every element in B is the image of some element in A; or equivalently, $Im f = B$.

The function in Example 3 is not onto either, because no negative real number in the co-domain is an image under f. Are the functions in Examples 1 and 2 onto ?

A function $f: A \to B$ is said to be *bijective* iff f is both one-one and onto.

EXAMPLE 4. Let $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x - 1$ for x in $\mathbf{R}$. This is both one-one and onto (why?) and so f is bijective.

EXAMPLE 5. Let $f: \mathbf{N} \to \mathbf{N}$ be given by $f(x) = x^2$ for all x in $\mathbf{N}$.

This is one-one but not onto. So f is not bijective. Note that in Example 3 we had the same defining relation for f, namely $f(x) = x^2$, but, there f was not one-one. So the defining relation alone does not determine one-one-ness of f.

EXAMPLE 6. Let $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = \sin x$ for all x in $\mathbf{R}$.

This is called the sine function. Here x may be taken to be in degrees or better in radians. Thus the relations $\sin 0 = 0$, $\sin \pi/6 = 1/2$, $\sin(-\pi/3) = -\sqrt{3}/2$ describe the images of 0, $\pi/6$ and $-\pi/3$ under the sine function. Here $Im f = [-1, 1]$ and so f is not onto. Also f is not one-one either as $f(0) = f(\pi) = f(2\pi) = \ldots = f(n\pi) = \ldots = 0$.

Similarly we have the cosine function, tangent function and so on. Note that for the tangent function, the domain is not all of $\mathbf{R}$. In fact, the function describes the tangent

$$f: \mathbf{R} - \{(2n+1)\frac{\pi}{2} \mid n \in Z\} \to \mathbf{R}, f(x) = \tan x$$

function. The tangent function is not defined at odd multiples of $\pi/2$ and so these are excluded from the domain. Also $Im f = \mathbf{R}$ and so the function is onto. Is it one-one ?

As stated before, our main interest is in functions defined on $\mathbf{R}$ or intervals and taking values in $\mathbf{R}$. Such functions are called *real-valued functions of a real variable.* They are of the form $f: X \to \mathbf{R}$, $X \subset \mathbf{R}$. These functions can be represented in the xy – plane by curves or graphs and usually just by looking at the graph of a real function, we can obtain a lot of information. The graph of a function $f: X \to \mathbf{R}$, $X \subset \mathbf{R}$ is the set of points $\{(x, f(x) \mid x \in X\}$ in the xy – plane. The relation $y = f(x)$ is the equation of the graph.

EXAMPLE 7. Let $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$, for all x in $\mathbf{R}$.

The equation to the graph of this function is $y - x^2$ and the graph has already been shown in Figure 5.3 of Chapter 5.

Suppose we draw a line parallel to x-axis and it cuts the graph of $y = f(x)$ in more than one point. What do we conclude? We conclude that f is not one-one. Thus f in Example 7 is not one-one, since every line parallel to the x-axis and lying above it cuts the graph in two distinct points (x_1, x_1^2) and $(-x_1, x_1^2)$.

How do we get an idea of the range of a function from its graph? Let us take the projection of the curve in Example 7 on the y-axis and examine which part of the y-axis is covered by this projection. We see that the whole of the positive y-axis and the origin are covered by this projection and so we conclude that $Im\, f = [0, \infty)$. So f is not onto.

EXAMPLE 8. Let us draw the graph of the sine function given in Example 6. The equation of the graph is $y = \sin x$ and the graph itself is given by Fig. 6.9.

The sine curve looks like a wave extending in either direction of the x-axis to infinity and does not go above $y = 1$ or below $y = -1$. This is because the range of the function is $[-1, 1]$. The function is neither one-one nor onto.

EXAMPLE 9. Let us draw the graph of the tangent function given by $f : \mathbf{R} - \{(2n-1)\pi/2 \mid n \in \mathbf{Z}\} \to \mathbf{R}, f(x) = \tan x$.

The equation of the graph is $y = \tan x$ and the graph is as given in Fig. 6.10.

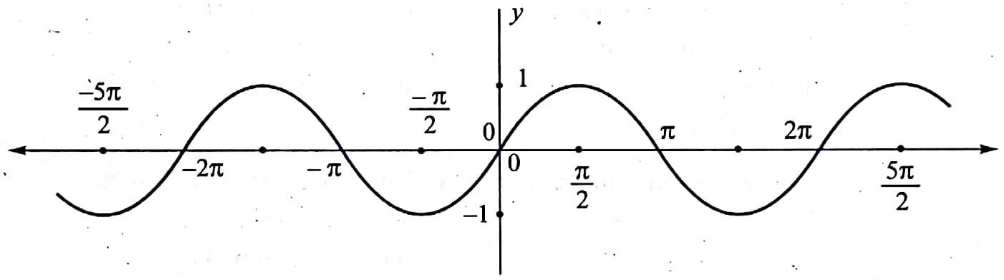


Fig. 6.9

The graph consists of infinitely many disjoint branches each lying between the two lines

$$x = \frac{(2n-1)\pi}{2},\ x = \frac{(2n+1)\pi}{2},\ n = 0, \pm 1, \pm 2, \ldots..$$

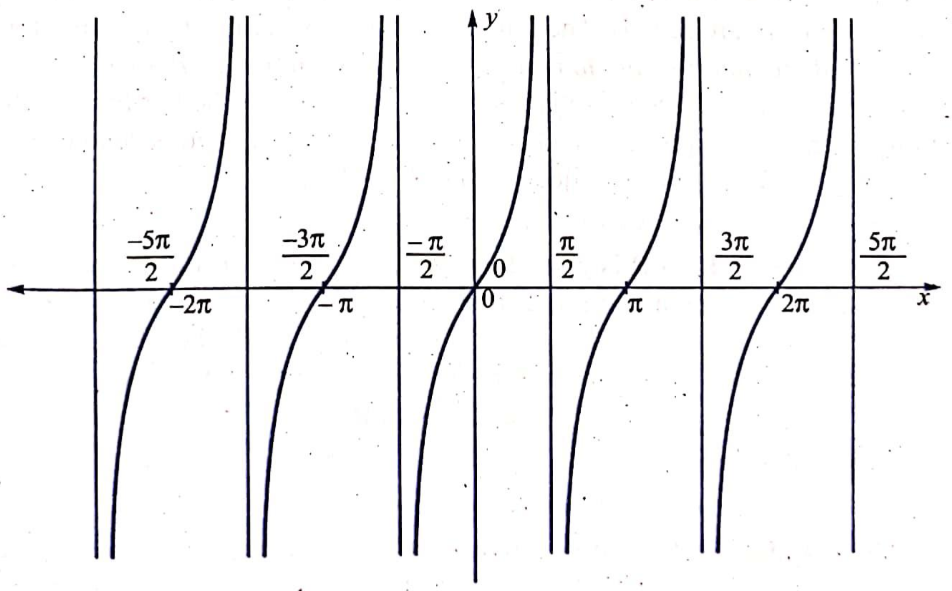


Fig. 6.10

These straight lines themselves are not a part of the graph. The points

$$x = \frac{(2n+1)}{2}\pi,\ n \in \mathbf{Z}$$

are precisely the points at which the graph is not only discontinuous but at which the function is not defined at all.

Table 6.5 gives the variations of the six trigonometric functions in the four quadrants. With the help of this table one can draw the graphs of the remaining trigonometric functions (see the exercises).

TABLE 6.5

As θ increases from π/2 to π,	**As θ increases from 0 to π/2,**
sin θ decreases from 1 to 0	sin θ increases from 0 to 1
cos θ decreases from 0 to –1	cos θ decreases from 1 to 0
tan θ increases from –∞ to 0	tan θ increases from 0 to ∞
cot θ decreases from 0 to –∞	cot θ decreases from ∞ to 0
sec θ increases from –∞ to –1	sec θ increases from 1 to ∞
cosec θ increased from 1 to ∞	cosec θ decreases from ∞ to 1
As θ increases from π to 3π/2,	**As θ increases from 3π/2 to 2π,**
sin θ decrease from 0 to –1	sin θ increases from –1 to 0
cos θ increases from –1 to 0	cos θ increases from 0 to 1
tan θ increases from 0 to ∞	tan θ increases from –∞ to 0
cot θ decreases from ∞ to 0	cot θ decreases from 0 to –∞
sec θ decreases from –1 to –∞	sec θ decreases from ∞ to 1
cosec ∞ increases from –∞ to –1	cosec θ decreases from –1 to –∞

We recall that if f is a bijective function from A to B, then we can define another function g from B to A in a natural way such that $gof : A \to A$ and fog : $B \to B$ are the identity functions on A and B respectively. (For the definition of the composite function fog, see No. 7 of Exercise 6.4). The function $I_X : X \to X$ defined by $I_x(x) = x$ for every $x \in X$ is called the *identity function* on X. In fact, if we define $g : B \to A$ by $g(y) = x$ iff $f(x) = y$, for each y in B, then g satisfies $gof = I_A$ and $fog\ g = I_B$. Further g is the only function which has this property and is called the *inverse* of f and is denoted by f^{-1}. It is seen that $f^{-1} : B \to A$ is also bijective and $(f^{-1})^{-1} = f$.

Also, $\qquad f(x) = y$ iff $f^{-1}(y) = x$.

EXAMPLE 10. If $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = 2x - 1$ for each x in $\mathbf{R}$, then it is easily verified that f is one-one and onto. The procedure by which we verify that f is onto generally gives the expression for f^{-1}. If $f(x) = y$ for y in $\mathbf{R}$, then $2x - 1 = y$ and so $x = (y + 1)/2$. Since $(y + 1)/2$ is a real number, we see that f is onto. Also f^{-1} is given by

$$f^{-1}(y) = (y + 1)/2 \text{ for each } y \text{ in } \mathbf{R}.$$

This may be rewritten as

$$f^{-1}(x) = (x + 1)/2 \text{ for each } x \text{ in } \mathbf{R}.$$

Now draw the graphs of f and f^{-1}, that is, graphs corresponding to the equations $y = f(x)$ and $y = f^{-1}(x)$. What do we observe? We see that the graphs are mirror images of each other relative to the line $y = x$. This is always the case.

EXAMPLE 11. Let $f: \mathbf{R} \to (0, \infty)$ be given by $f(x) = 2^x$, for each x in $\mathbf{R}$. Observe that f is a bijective function. The inverse function $f^{-1}: (0, \infty) \to \mathbf{R}$ is given by

$$f^{-1}(x) = \log_2 x, \text{ for } x \text{ in } (0, \infty).$$

Draw the graphs of these functions and observe that they are reflections of each other with respect to the line $y = x$.

Note. The *logarithmic function* introduced in this example is an important function in Mathematics and its applications. When the base is 10, instead of 2 as in Example 11, the corresponding logarithmic function is written as simply, $\log x$. When the student goes to higher levels of Mathematics he will see the need to write it fully as $\log_{10} x$.

Remark 1. Suppose we have a function $f: A \to B$ which is one-one but not necessarily onto. Can we produce a function g which is the same as f for all practical purposes but at the same time has the additional property that g is onto, besides being one-one? Yes. Simply delete the elements in B which are not, images under f and retain the function as it is. In precise terms, $g : A \to f(A)$ given by $g(x) = f(x)$ for each x in A, has the property that g is bijective. This is exactly what we have done in Example 11, to modify the function $f: \mathbf{R} \to \mathbf{R}, f(x) = 2^x$ into a bijective function $f: \mathbf{R} \to (0, \infty)$.

Further suppose that $f: A \to B$ is neither one-one nor onto. As above f can be made onto by deleting elements of B which are not images. How can we make f one-one also? There are two ways. One way is to define f on a suitable partition of A, but not on A. This altogether alters the domain of f. Another way is to delete elements of A (in a suitable way) to make the function one-one as in the following examples.

The function $f: \mathbf{R} \to \mathbf{R}, f(x) = x^2$ for each x in $\mathbf{R}$ is made onto by deleting $(-\infty, 0)$ from the co-domain and one-one by deleting $(-\infty, 0)$ from the domain. We obtain the function

$$f: [0, \infty) \to [0, \infty),$$

$$f(x) = x^2, \text{ for each } x \text{ in } [0, \infty).$$

(We retain the same name f of the function).

The inverse of f is given by

$$f^{-1}: [0, \infty) \to [0, \infty)$$

$$f^{-1}(x) = \sqrt{x}, \text{ for each } x \text{ in } [0, \infty).$$

This approach helps us to define the inverses of trigonometric functions [see Section 6].

Sometimes the dependent variable y is not explicitly given by a function of the independent variable, but they are related by an equation. This gives rise to what are called *implicit functions*.

Consider the equation

$$x^2 + y^2 = 4$$

This is the equation of a circle with center (0, 0) and radius 2. The graph is as given in Figure 6.11. The circle meets the x-axis in (2, 0), (–2, 0) and the y-axis in (0, 2) and (0, – 2). The eight points (± 6/5, ± 8/5), (± 8/5, ± 6/5) are also on the circle. The circle here is the collection of points namely $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 4\}$.

From the graph, we see that the equation $x^2 + y^2 = 4$ does not define a function as there are two images for any point x in (–2, 2). This is also clear from the equation itself, for, if we solve for y, we get

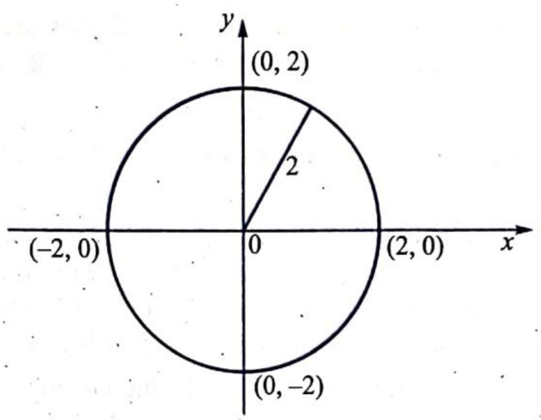


Fig. 6.11

$$y = \pm\sqrt{(4 - x^2)}\,.$$

Thus y has two values for each x in $(-2, 2)$.

Nevertheless, we have a 'function-like' structure. In fact, the graph is the union of the two graphs given by the functions

$$f : [-2, 2] \to \mathbf{R}, f(x) = \sqrt{4 - x^2}$$

and $$g : [-2, 2] \to \mathbf{R}, g(x) = -\sqrt{4 - x^2}\,.$$

Here f corresponds to the upper semicircle and g to the lower semicircle.

Observe that a **graph is symmetric with respect to the y-axis if its equation is unaltered by replacing x by $-x$** (e.g., $y = x^2$, $y = x^4$). **Similarly a graph is symmetric with respect to the x-axis if its equation is unaltered by replacing y by $-y$.** The graph of the circle above in Example 13 is symmetric with respect to both the axes.

EXERCISE 6.4

1. (*a*) Determine which of the figures from Fig. 6.12 to Fig. 6.15 represent functions from A to B ?

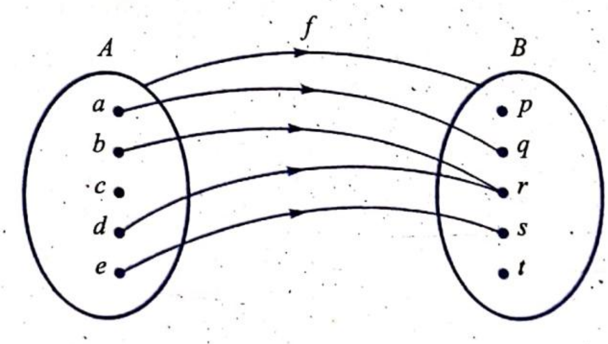


Fig. 6.12

(*b*) Which f is a function ?

(*i*) $f : \mathbf{Z} \to \mathbf{N}, f(x) = x^2$, for all x in $\mathbf{Z}$.

(*ii*) $f : \mathbf{Z} \to \mathbf{Z}, f(x) = x^2$, for all x in $\mathbf{Z}$.

(*iii*) $f: \mathbf{N} \to \mathbf{R}$, $f(x) = \dfrac{1}{2x^2 - 7x - 15}$, for all x in **N**.

(*iv*) $f: \mathbf{R} \to \mathbf{R}$, $f(x) = \dfrac{1}{x^2 + x + 1}$, for all x in **R**.

2. Find the inverses of the following functions whenever they exist.

(*a*) $f: \mathbf{R} \to \mathbf{R}, f(x) = x^3$, for all x in **R**.

(*b*) $f: \mathbf{R} \to \mathbf{R}, f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$

(*c*) $f: \mathbf{R} \to \mathbf{R}, f(x) = |x|$, for all x in **R**.

(*d*) $f: \mathbf{R} \to \mathbf{R}, f(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$

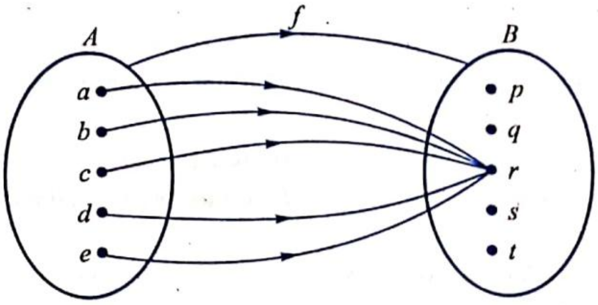


Fig. 6.13

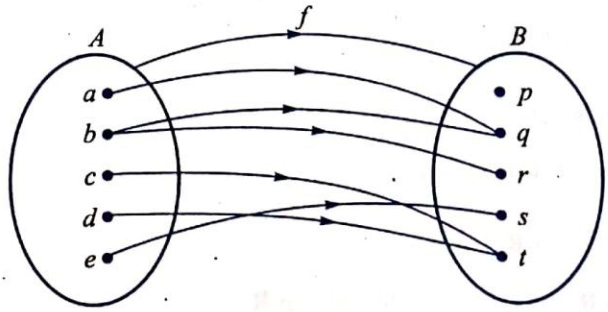


Fig. 6.14

(*e*) $f: (-\pi/2, \pi/2) \to \mathbf{R}, f(x) = \tan x$, for all x in $(-\pi/2, \pi/2)$

(*f*) $f: \mathbf{N} \to \mathbf{N}, f(n) = n^2 + 9n + 14$, for all n in *IN*.

(*g*) $f: \mathbf{N} \to \text{N}, f(n) = \begin{cases} n+1, \text{if } n \text{ is odd}, \\ n-1, \text{if } n \text{ is even}. \end{cases}$

(*h*) $f: \mathbf{R} - \{0\} \to \mathbf{R}\, f(x) = 1/x$

3. Draw the graphs of the following

(*a*) $y = \cos x$.
(*b*) $y = \cot x$.
(*c*) $y = \operatorname{cosec} x$.
(*d*) $y = \sec x$.
(*e*) $y = \sin 2x$,
(*f*) $y = \tan^2 x$.
(*g*) $y = |x|$.

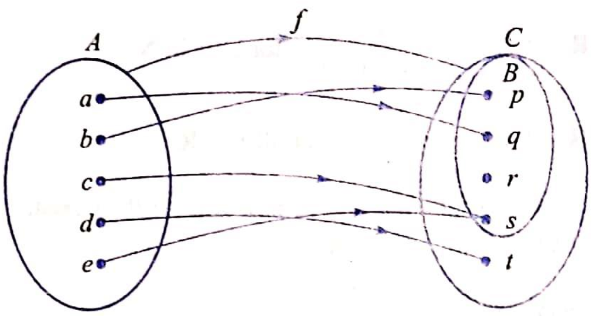


Fig. 6.15

4. Find the largest domains, as subsets of **R**, of the following functions.

(*a*) $f(x) = \log |x|$.
(*b*) $f(x) = \dfrac{1}{\log |x|}$

(*c*) $f(x) = \sqrt{9 - x^2}$.
(*d*) $f(x) = 1/\sqrt{x^2 - 9}$

(*e*) $f(x) = \log \sin x$.
(*f*) $f(x) = \log |\sin x|$.

5. Draw the graphs of

(*a*) $|x| + |y| = 1$.
(*b*) $\max\{|x|, |y|\} = 1$.

(*c*) $|x - y| + x + y = 2$.
(*d*) $y = ax^2 + bx + c, a \neq 0$.

(*e*) $y - |\sin x|$.

(**Note** For (*d*) consider the five cases $a > 0$, $a < 0$ and $b^2 - 4ac <, >$ or $= 0$ separately).

6. How are the graphs of the following equations related to one another?

(*a*) $y^2 = x,\ y = \sqrt{x},\ y = -\sqrt{x},\ |y| = \sqrt{x}$.

(*b*) $y = x^3, y = x^{1/3}$.

(*c*) $y = x^3, |y| = |x|^3, |y| = x^3, y = |x|^3$.

(*d*) (*i*) $f : \mathbf{R} \to \mathbf{R},\ f(x) = \dfrac{x^2 - 1}{x - 1}, x \neq 1$

and $f(x) = 2$, when $x = 1$.

(*ii*) $f : \mathbf{R} - \{1\} \to \mathbf{R},\ f(x) = \dfrac{x^2 - 1}{x - 1}$,

(*iii*) $f : \mathbf{R} \to \mathbf{R},\ f(x) = x + 1$, for all x in **R**

(*iv*) $f : \mathbf{R} \to \mathbf{R},\ f(x) = \dfrac{x^2 - 1}{x - 1}, x \neq 1$,

and $f(x) = 1$ when $x = 1$.

7. If $f : A \to B$, $g : B \to C$ are two functions then *the composite function* $gof : A \to C$ is defined by $(gof)(x) = g(f(x))$ for all x in A. Show that

(*i*) if f and g are one-one, then so is gof;

(*ii*) if f and g are onto, then so is gof;

(*iii*) if f and g are bijective, so is gof;

(*iv*) if f and g are bijective, then $(gof)^{-1} = f^{-1} o g^{-1}$.

8. If $f : A \to B$ is a function, then f is said to be a *constant* function iff $Im f$ is a singleton set, that is, iff $f(x) = y_0$ for all x in A and for some fixed y_0 in B. Draw the graph of the function

$f : \mathbf{R} \to \mathbf{R}, f(x) = k$, for all x in **R**, k being a fixed real constant.

9. Suppose A and B are two finite sets having the same cardinality, that is, having the same number of elements. Show that a function $f : A \to B$ is one-one iff it is onto.
10. Let A and B be two finite sets having m elements and n elements respectively ($m \neq 0$, $n \neq 0$).
    (a) Find the number of all functions from A to B;
    (b) Find the number of one-one functions from A to B ($m \leq n$);
    (c) For $n = 2, 3$ find the number of onto functions from A to B ($m \geq n$).
11. From the graphs of the sine, cosine and tangent functions, show that the general solution of $\sin x = 0$ or $\tan x = 0$ is given by $x = n\pi$, where n is any integer and that of $\cos x = 0$ is given by
    $x = (2n + 1)\pi/2$, where n is any integer.

6.5 A : RATIOS OF COMPOUND ANGLES

If $A, B, C,...$ are any angles, then expressions such as $A + B, A - B, 2A - 3B + C, 90° - A$ are called *compound angles*. In this section we find expressions for ratios of $A + B$, $A - B$, $2A$ and $3A$ in terms of ratios of A and B, which will help us handle the ratios of other compound angles.

Theorem 1. If A and B are two angles, then

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$;

(b) $\cos(A + B) = \cos A \cos B - \sin A \sin B$;

(c) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Proof. We shall consider only the case in which A, B and $A + B$ lie between 0 and 90°. See Fig. 6.16. In the figure $\angle XOY = A$, $\angle YOZ = B$ both traced in the positive direction, so that $\angle XOZ = A + B$. On ray $\overrightarrow{OZ}$, choose a point P. Draw PQ perpendicular to $\overrightarrow{OX}$, PR perpendicular to $\overrightarrow{OY}$, RS perpendicular to $\overrightarrow{OX}$, and RT perpendicular to PQ. Clearly $QSRT$ is a rectangle, so that $QS = TR$ and $TQ = RS$. Also $\angle QPR = \angle XOY = A$ as PQ and PR are respectively perpendicular to OX and OY.

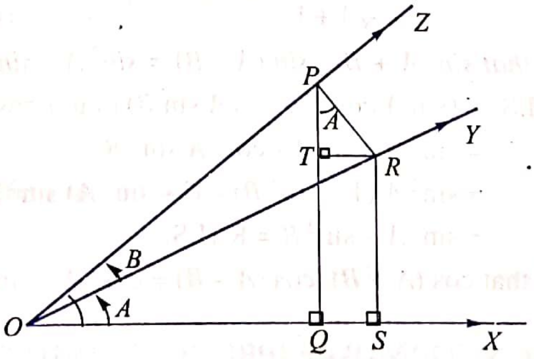


Fig. 6.16

(a) From triangle OPQ, in which $\angle QOP = A + B$,

$$\sin(A + B) = \frac{PQ}{OP} = \frac{PT + TQ}{OP} = \frac{PT}{OP} + \frac{RS}{OP}$$

$$= \frac{PT}{PR} \cdot \frac{PR}{OP} + \frac{RS}{OR} \cdot \frac{OR}{OP} = \cos A \sin B + \sin A \cos B.$$

(*b*) Again, $\cos(A + B) = \frac{OQ}{OP} = \frac{OS - QS}{OP} = \frac{OS}{OP} - \frac{TR}{OP} = \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$

$$= \cos A \cos B - \sin A \sin B.$$

(*c*) Further, to prove the last relation, we observe from (*a*) and (*b*) that

$\sin (A + B) = \cos A \cos B (\tan A + \tan B)$; and

$\cos (A + B) = \cos A \cos B (1 - \tan A \tan B)$.

Dividing the first relation by the second we obtain the result. ❑

Theorem 2. If A and B are any two angles then

(*a*) $\sin (A - B) = \sin A \cos B - \cos A \sin B$;

(*b*) $\cos (A - B) = \cos A \cos B + \sin A \sin B$;

(*c*) $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

Proof. One has only to replace B by $-B$ in Theorem 1 and observe that $\sin (-B) = -\sin B$, $\cos (-B) = \cos B$ and $\tan (-B) -\tan B$. ❑

EXAMPLE 1. $\sin (\theta + \pi/4) = \sin \theta \cos \pi/4 + \cos \theta \sin \pi/4$

$$= (1/\sqrt{2}) (\sin \theta + \cos \theta).$$

EXAMPLE 2. $\cos (2A + B/3) = \cos 2A \cos B/3 - \sin 2A \sin B/3$.

EXAMPLE 3. $\tan (\pi/4 + \theta) = \frac{1 + \tan \theta}{1 - \tan \theta}$.

EXAMPLE 4. $\sin (\pi/4 - \theta) = \sin (\pi/4) \cos \theta - \cos (\pi/4) \sin \theta$

EXAMPLE 5. $\tan 15° = \tan (60° - 45°) = \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}.$$

EXAMPLE 6. *Show that* $\sin (A + B) . \sin (A - B) = \sin^2 A - \sin^2 B$.

SOLUTION. L.H.S. $= (\sin A \cos B + \cos A \sin B) (\sin A \cos B - \cos A \sin B)$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$$

$$= \sin^2 A - \sin^2 B = \text{R.H.S.}$$

EXAMPLE 7. Prove that $\cos (A + B). \cos (A - B) = \cos^2 A - \sin^2 B$.

6.5 B : CONVERSION FORMULAE (PRODUCTS INTO SUMS)

Using the formulae for $\sin (A + B)$, $\sin (A - B)$, $\cos (A + B)$ and $\cos (A - B)$, one easily deduces the following important conversion formulae:

Theorem 3. If A and B are two angles,

(*a*) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$,

(*b*) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$,

(*c*) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$,

(*d*) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

To prove these we start from the R.H.S. of each equation and obtain the corresponding L.H.S.

Note. Formulae (*a*) and (*b*) are essentially the same; for, we have only to interchange *A* and *B*.

EXAMPLE 8. $2 \sin 80° \cos 20° = \sin (80° + 20°) + \sin (80° - 20°)$

$$= \sin 100° + (\sqrt{3}/2) = \sin 80° + (\sqrt{3}/2)$$

EXAMPLE 9. $2 \sin 10° \sin 50° = \cos (50° - 10°) - \cos (50° + 10°)$

$$= \cos 40° - (1/2).$$

EXAMPLE 10. $2 \cos \left(\frac{A+3B}{2}\right) \cos \left(\frac{3A-B}{2}\right) = \cos(2A + B) + \cos(A - 2B)$

EXAMPLE 11. $\sin (A + B) . \sin (A - B) = \frac{1}{2} [2\sin (A + B). \sin (A - B)]$

$$= \frac{1}{2} (\cos 2B - \cos 2A).$$

6.5 C : CONVERSION FORMULAE (SUMS INTO PRODUCTS)

We have another set of four more important formulae which express sums (and differences) as products. They are given in the following Theorem.

Theorem 4. If *C* and *D* are two angles,

(*a*) $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$;

(*b*) $\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$;

(*c*) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$;

(*d*) $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$;

Note that the last angle is $(D - C)/2$ and not $(C - D)/2$. These are proved by writing $A = (C + D)/2$ and $B = (C - D)/2$ in Theorem 3. ❑

EXAMPLE 12. $\sin 70° + \sin 10° = 2 \sin \frac{70° + 10°}{2} \cos \frac{70° - 10°}{2}$

$$= 2 \sin 40° \cos 30° = \sqrt{3} \sin 40°.$$

EXAMPLE 13. $\cos 20° - \sin 20° = \cos 20° - \cos 70°$

$$= 2 \sin \frac{20° + 70°}{2} \sin \frac{70° - 20°}{2}$$

$$= 2 \sin 45° \cos 25° = \sqrt{2} \sin 25°.$$

EXAMPLE 14. $\sin 7\theta - \sin 3\theta = 2 \sin \frac{7\theta - 3\theta}{2} \cos \frac{7\theta + 3\theta}{2}$

$$= 2 \sin 2\theta \cos 5\theta.$$

EXAMPLE 15. *Show that*

$$\frac{\sin A + 2\sin 5A + \sin 9A}{\cos A + 2\cos A + \cos 9A} = \tan 5A.$$

SOLUTION. $\text{L.H.S.} = \dfrac{(\sin A + \sin 9A) + 2\sin 5A}{(\cos A + \cos 9A) + 2\cos 5A}$

$$= \frac{2\sin 5A\cos 4A + 2\sin 5A}{2\cos 5A\cos 4A + 2\cos 5A}$$

$$= \frac{2\sin 5A\,.\,(\cos 4A + 1)}{2\cos 5A\,.\,(\cos 4A + 1)}$$

$$= \tan 5A = \text{R.H.S.}$$

6.5 D : RATIOS OF MULTIPLE ANGLES

More formulae ! But the more the formulae, the better is the facility in handling involved expressions. If A is an angle and n is a positive integer we say nA is a multiple of A and $(1/n)\,A$ is a submultiple of A. Thus $2A$, $3A$,.. are multiples of A and $(1/2)\,A$, $(1/3)\,A$,... are submultiples of A.

Theorem 5. If A is any angle, then

(*a*) $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$;

(*b*) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$$= 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A};$$

(*c*) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

Proof. (*a*) Put $B = A$ in $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

We get $\sin(A+A) = \sin A\cos A + \cos A\sin A$.

That is, $\sin 2A = 2\sin A\cos A$

Further, $2\sin A\cos A = \dfrac{2\sin A}{\cos A}\cdot\cos^2 A$

$$= 2\tan A\,\frac{1}{\sec^2 A} = \frac{2\tan A}{1+\tan^2 A}.$$

Similarly (*b*) and (*c*) are proved by putting $B = A$ in the expressions for $\cos(A+B)$ and $\tan(A+B)$.

Replacing A by $A/2$ in the above formulae we can express ratios of A in terms of ratios of $A/2$. Thus

(*i*) $\sin A = 2\sin(A/2)\cos(A/2) = \dfrac{2\tan(A/2)}{1+\tan^2(A/2)}$.

(*ii*) $\cos A = \cos^2(A/2) - \sin^2(A/2) = 2\cos^2(A/2) - 1 = 1 - 2\sin^2(A/2)$

$$= \frac{1 - \tan^2(A/2)}{1 + \tan^2(A/2)}.$$

(*iii*) $\tan A = \dfrac{2\tan(A/2)}{1 + \tan^2(A/2)}$. □

EXAMPLE 16. $\sin 4\theta = 2\sin(4\theta/2)\cos(4\theta/2) = 2\sin 2\theta \cos 2\theta$.

EXAMPLE 17. $\cos(A+B) = 2\cos^2\dfrac{(A+B)}{2} - 1 = \dfrac{1 - \tan^2((A+B)/2)}{1 + \tan^2((A+B)/2)}$.

EXAMPLE 18. $\tan 40° = \dfrac{2\tan 20°}{1 - \tan^2 20°}$.

Theorem 6. If A is any angle, then

(*a*) $\sin 3A = 3\sin A - 4\sin^3 A$;

(*b*) $\cos 3A = 4\cos^3 A - 3\cos A$;

(*c*) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

Proof. (*a*) $\sin 3A = \sin(2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A(1 - \sin^2 A) + \sin A(1 - 2\sin^2 A)$

$= 3\sin A - 4\sin^3 A$.

Aliter $\sin 3A - \sin A = 2\sin\dfrac{3A - A}{2}\cos\dfrac{3A + A}{2}$

$= 2\sin A\cos 2A = 2\sin A.(1 - 2\sin^2 A)$

$= 2\sin A - 4\sin^3 A$.

Hence $\sin 3A = 3\sin A - 4\sin^3 A$.

The other formulae are similarly proved. □

EXAMPLE 19. $\sin A = \sin 3(A/3) = 3\sin(A/3) - 4\sin^3(A/3)$.

EXAMPLE 20. $\dfrac{1}{2} = \cos 60° = 4\cos^3 20° - 3\cos 20°$.

EXAMPLE 21. $\tan(A+B+C) = \dfrac{3\tan\frac{A+B+C}{3} - \tan^3\left(\frac{A+B+C}{3}\right)}{1 - 3\tan^2\left(\frac{A+B+C}{3}\right)}$

EXAMPLE 22. *Show that*

$$4\sin\theta\sin(\pi/3 + \theta)\sin(\pi/3 - \theta) = \sin 3\theta.$$

SOLUTION. L.H.S. $= 4\sin\theta[\sin^2\pi/3 - \sin^2\theta]$

$= 4\sin\theta(3/4 - \sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$

$= \sin 3\theta =$ R.H.S.

6.5 E : RATIOS OF 18° AND 36° :

Let $\theta = 18°$, so that $5\theta = 90°$ and $2\theta = 90° - 3\theta$.

This gives $\sin 2\theta = \cos 3\theta$; that is $2 \sin\theta \cos\theta = 4 \cos^3\theta - 3 \cos \theta$

Dividing by $\cos \theta$ ($\neq 0$), we get

$$2 \sin \theta = 4 \cos^2 \theta - 3 = 1 - 4 \sin^2 \theta.$$

Therefore, $4 \sin^2 \theta + 2 \sin \theta - 1 = 0$. Solving this for $\sin \theta$, we obtain

$$\sin \theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Since $\theta = 18°$, $\sin \theta$ is positive.

Therefore have,

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

Now the other ratios of 18° can be found out, as also those of 36°, 72° and 54°. For example, $\cos 72° = \sin 18° = (\sqrt{5} - 1)/4$; and so on.

Now we give geometrical proofs of the expressions for sin 18° and cos 36°. Consider an isosceles triangle ABC in which $AB = AC$ and $\angle A = 36°$, so that $\angle B = \angle C = 72°$. Let the internal biector of $\angle C$ meet AB in D. Since $\angle BCD = \angle ACD = 36°$, we have $\angle BDC = 72°$. So BCD is an isosceles triangle and $BC = DC$. Again $\angle DAC = \angle DCA = 36°$ and so triangle ACD is isosceles in which $AD = DC$. Draw AE and DF perpendicular to BC and AC respectively. Let $AB = AC = a$ and $BC = DC = DA = x$. Now triangle ABC is similar to triangle CDB. Hence

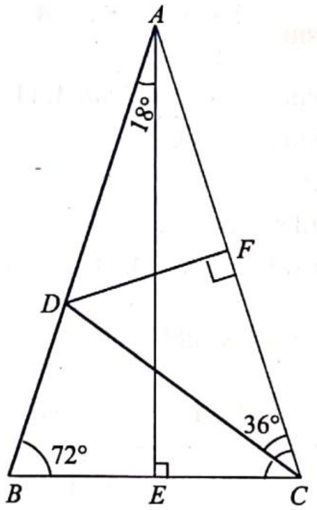


Fig. 6.17

$$AB/CD = BC/DB.$$

That is, $a/x = x/(a - x)$. Simplifying we get $x^2 + ax - a^2 = 0$.

Solving we have $x/a = (-1 + \sqrt{5})/2$, as negative sign is ruled out.

In triangle ABE, $\angle BAE = 18°$, $\therefore$ $\sin \angle BAE = BE/AB$.

That is, $\sin 18° = \dfrac{x/2}{a} = \dfrac{1}{2}(x/a) = (\sqrt{5} - 1)/4.$

Also from triangle DAF, $\cos \angle DAF = AF/AD$.

That is, $\cos 36° = \dfrac{a/2}{x} = \dfrac{1}{2}(a/x) = \dfrac{1}{2}\dfrac{2}{(\sqrt{5}-1)} = \dfrac{\sqrt{5}+1}{4}.$

EXAMPLE 23. $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°.$

EXAMPLE 24. $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°.$

EXAMPLE 25. *Show that 2 sin 48° sin 12° = sin 18°.*

SOLUTION. L.H.S $= \cos 36° - \cos 60°$

$$= \frac{\sqrt{5}+1}{4} - \frac{1}{2} = \frac{\sqrt{5}-1}{4} = \sin 18° = \text{R.H.S.}$$

EXERCISE 6.5

1. Show that $\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$.
2. Show that $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$.
3. Show that

   (*a*) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

   (*b*) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

   (*c*) $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$.

4. Expand $\sin(A+B+C)$, $\cos(A+B+C)$, $\tan(A+B+C)$ in terms of the ratios of A, B, C.
5. (*a*) Show that

   $$\tan(A_1 + A_2 + \dots + A_n) = \frac{s_1 - s_3 + s_5 - s_7 + \dots}{1 - s_2 + s_4 - s_6 + \dots}.$$

   where s_r = sum of products of tangents of the angles $A_1, A_2, \dots, A_n$ taken r at a time. (Use induction).

Prove the following identities [(6) – (12)]:

6. $\cos 3\theta = 4\cos\theta \cos(\theta - \pi/3)\cos(\theta + \pi/3)$.
7. $\tan 3\theta = \tan\theta \tan(\pi/3 - \theta)\tan(\pi/3 + \theta)$.
8. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.
9. $(\sin 8\theta)/\sin\theta = 8(16\cos^7\theta - 24\cos^5\theta + 10\cos^3\theta - \cos\theta)$.
10. $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$.
11. $1 + \tan A \tan(A/2) = \tan A \cot(A/2) - 1 = \sec A$.

12. (*a*) $\cot A - \tan A = 2 \cot 2A$.

(*b*) Hence express $\tan A + 2 \tan 2A + 2^2 \tan 2^2 A + \ldots + 2^{n-1} \tan 2^{n-1} A$ in the form $a \cot A + b \cot 2^n A$, where a and b are integers.

Prove the following relations (13) – (20):

13. $\cos 20° \cos 40° \cos 60° \cos 80° = 1/16$.

14. $\sin 20° \sin 40° \sin 60° \sin 80° = 3/16$.

15. $\cos^2 74° + \cos^2 14° - \cos 74° \cos 14° = 3/4$.

16. (*a*) $\tan (7\frac{1}{2})° = \sqrt{6} - \sqrt{4} - \sqrt{3} + \sqrt{2}$.

(*b*) $\cot 15° = 2 + \sqrt{3}$.

17. $\cot 20° - \cot 40° + \cot 80° = \sqrt{3}$.

18. $\tan 6° \tan 42° \tan 66° \tan 78° = 1$.

$$\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cdot \cos \frac{3\pi}{15} \ldots \cos \frac{7\pi}{15} = 1/2^7.$$

$\cos\theta + \cos 5\theta + \cos 7\theta = \cos 2\theta + \cos 4\theta$, where $\theta = 12°$.

Show that $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.

Show that $4 \sin^3 \theta \cos 3\theta + 4 \cos^3 \theta \sin 3\theta = 3 \sin 4\theta$.

If $\cos \alpha + \cos \beta = p$ and $\sin \alpha + \sin \beta = q$, evaluate $\tan^2 (\alpha - \beta)/2$ and $\tan^2 (\alpha - \beta)/2$ in terms of p and q.

If $\cos \theta = \dfrac{\cos u - e}{1 - e \cos u}$, then show that

$$\tan (\theta/2) = \pm \sqrt{\frac{1+e}{1-e}} \tan (u/2).$$

If $\text{cosec} (\theta + \alpha) \text{cosec} (\theta - \alpha) = 2 \text{cosec} \theta$ then show that either $\alpha = 2n\pi$, where n is an integer or $\sin \theta = \pm \sqrt{2} \cos (\alpha/2)$.

28. For what values of A in $[0, 2\pi]$ does the following relation hold good?

(*a*) $2 \sin(A/2) = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$;

(*b*) $2 \cos(A/2) = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}$.

Use the idea given in exercise 26 to

show that $$\sin 9° = \frac{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}}{4}$$

and $$\cos 9° = \frac{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}}{4}.$$

Evaluate $\sin 27°$ and $\cos 27°$.

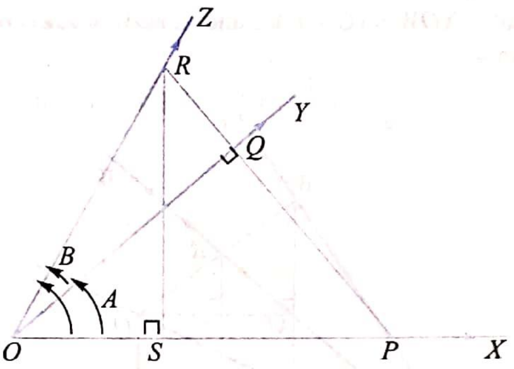
Find the ratios of 15° and 75°.

(*a*) In the adjoining figure (Fig. 18), $\angle XOY = A$, $\angle YOZ = B$.

The line PQR is perpendicular to $\overrightarrow{OY}$ and RS is perpendicular to $\overrightarrow{OX}$.

Use the fact that $2\Delta OPR = OP.RS - PR.OQ$ to prove that

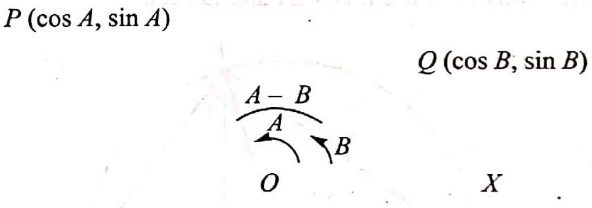
$\sin(A + B) = \sin A \cos B + \cos A \sin B.$



(*b*) Use Table 6.3, Section 6.3 to obtain expressions for $\sin(A - B)$, $\cos(A + B)$ and $\cos(A - B)$.

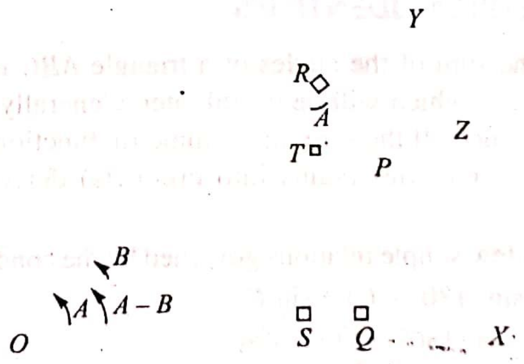
In the *xy*-plane consider the unit circle (centre 0, radius 1) and take two points $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$ on the circle as shown in Fig. 6.19. Use the distance formula for PQ in two ways to obtain

$\cos(A - B) = \cos A \cos B + \sin A \sin B.$



Give a purely geometrical proof of the formula for $\tan(A + B)$, using figure 6.16. (Use the similarity of triangles PTR and OSR).

In figure 6.20, $\angle XOY = A$, $\angle ZOY = B$ and PQ, PR, RS, PT are respectively perpendicular to OX, OY, OX, RS. Give geometrical proofs of the formulae in Theorem 2, that is, the expressions for the ratios of $A - B$. (Imitate the proof of Theorem 1).



34. In figure 6.21, $\angle XOY = C$, $\angle ZOY = D$, OW bisects $\angle YOZ$, the line PRQ is perpendicular to OW. The lines PS, QT, RL are all perpendicular to OX and the line QNM is perpendicular to PS. Observe that $\angle XOW = (C + D)/2$ and $\angle YOW = \angle ZOW = (C - D)/2$. Deduce the results of Theorem 4.

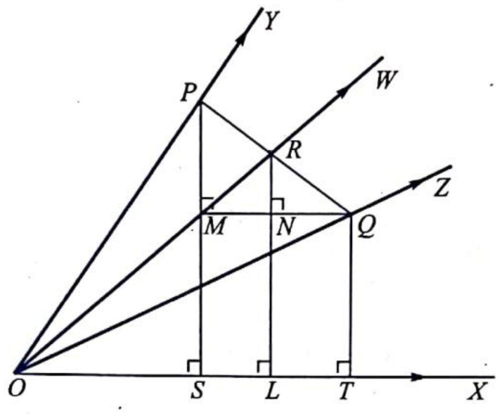


Fig. 6.21

35. Let BC be a diameter of a circle centre O and P a point on the circle such that $\angle CBP = A$, PQ perpendicular to BC as in figure 6.22. Observe that $\angle COP = 2A$ and $\angle CPQ = A$. Obtain the expressions for sin $2A$. cos $2A$ and tan $2A$.

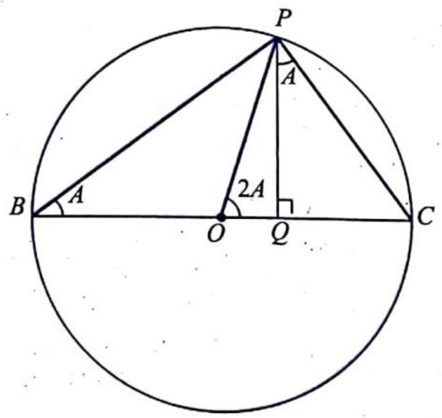


Fig. 6.22

6.6 TRIGONOMETRICAL IDENTITIES

Using the fact that the sum of the angles of a triangle ABC is 180 degrees, we can derive several identities, which will be useful later. Generally, in these identities we express sums into products. If the sums are symmetric functions of A, B, C, so are the products. The conversion rules (sums into products) derived in Section 6.5 are extensively used.

First we mention a few simple relations governed by the condition $A + B + C = 180°$:

(*i*) $\sin (A + B) = \sin (180° - C) = \sin C$.

(*ii*) $\cos (B + C) = \cos (180° - A) = -\cos A$.

(*iii*) $\tan (C + A) = \tan (180° - B) = -\tan B$.

(*iv*) $\sin((A + B)/2) = \sin(90° - C/2) = \cos(C/2)$.

(*v*) $\cos((B + C)/2) = \cos(90° - A/2) = \sin(A/2)$.

(*vi*) $\tan((C + A)/2) = \tan(90° - B/2) = \cot(B/2)$.

(*vii*) $\sin((A + B)/4) = \sin((\pi - C)/4) = \sin\left(\frac{\pi}{2} - \frac{\pi + C}{4}\right)$

$$= \cos((\pi + C)/4).$$

EXAMPLE 1. *If $A + B + C = 180°$ show that*

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$$

SOLUTION. We have

$$\begin{aligned}\cos 2A + \cos 2B + \cos 2C &= 2\cos(A + B)\cos(A - B) + \cos 2C\\ &= -2\cos C\cos(A - B) + 2\cos^2 C - 1\\ &= -1 - 2\cos C[\cos(A - B) - \cos C]\\ &= -1 - 2\cos C[\cos(A - B) + \cos(A + B)]\\ &= -1 - 2\cos C.\,2\cos A\cos B\\ &= -1 - 4\cos A\cos B\cos C.\end{aligned}$$

EXAMPLE 2. *If $A + B + C = \pi$, show that*

$$\sin A + \sin B - \sin C = 4\sin(A/2)\sin(B/2)\cos(C/2).$$

SOLUTION. We have

$$\begin{aligned}\sin A + \sin B - \sin C &= 2\sin(A + B)/2\cos(A - B)/2 - \sin C\\ &= 2\cos(C/2)\cos((A - B)/2) - 2\sin(C/2)\cos(C/2)\\ &= 2\cos(C/2)[\cos((A - B)/2) - \sin(C/2)]\\ &= 2\cos(C/2)[\cos((A - B)/2) - \cos((A + B)/2)]\\ &= 2\cos(C/2)\,2\sin(A/2)\sin(B/2)\\ &= 4\sin(A/2)\sin(B/2)\cos(C/2).\end{aligned}$$

EXAMPLE 3. *If the sum of the three angles A, B, C is 2 right angles, show that*

$$\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2) = 1 - 2\sin(A/2)\sin(B/2)\sin(C/2).$$

SOLUTION. We have

$$\begin{aligned}&\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\\ &= 1 - (\cos^2(A/2) - \sin^2(B/2)) + \sin^2(C/2)\\ &= 1 - \cos((A + B)/2)\cos((A - B)/2) + \sin^2(C/2)\\ &= 1 - \sin(C/2)\cos((A - B)/2) + \sin^2(C/2)\\ &= 1 - \sin(C/2)[\cos((A - B)/2) - \sin(C/2)]\\ &= 1 - \sin(C/2)[\cos((A - B)/2) - \cos((A + B)/2)]\\ &= 1 - \sin(C/2)\,2\sin(A/2)\sin(B/2)\\ &= 1 - 2\sin(A/2)\sin(B/2)\sin(C/2).\end{aligned}$$

Alternatively, one may use the formula

$$\sin^2\theta/2 = (1/2)(1 - \cos\theta)$$

and write the left hand expression in the form

$$3/2 - 1/2(\cos A + \cos B + \cos C),$$

and proceed with $\cos A + \cos B + \cos C$ as in Example 2.

EXAMPLE 4. *If $A + B + C$ is a multiple of π, then prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

SOLUTION. Let $A + B + C = n\pi$, where n is any integer. Then

$$A + B = n\pi - C.$$

Therefore $\tan (A + B) = \tan (n\pi - C)$.

So $$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C.$$

$\therefore \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C.$

That is, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. As a consequence, we see that if $A + B + C = \pi$, then $\tan nA + \tan nB + \tan nC = \tan nA.\tan nB.\tan nC$, for any integer n.

EXERCISE 6.6

If $A + B + C = 180°$, show that

1. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$
2. $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C.$
3. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$, and hence that $\cos 2A - \cos 2B - \cos 2C = -1 + 4 \cos A \sin B \sin C.$
4. $\sin A + \sin B + \sin C = 4 \cos (A/2) \cos (B/2) \cos (C/2).$
5. $\cos A + \cos B + \cos C = 1 + 4 \sin (A/2) \sin (B/2) \sin (C/2).$
6. $-\cos A + \cos B + \cos C = -1 + 4 \sin (A/2) \cos (B/2) \cos (C/2).$
7. $\sin^2(A/2) - \sin^2(B/2) + \sin^2(C/2) = 1 - 2 \cos (A/2) \sin (B/2) \cos (C/2).$
8. $\cos^2(A/2) + \cos^2(B/2) + \cos^2(C/2) = 2 + 2 \sin (A/2) \sin (B/2) \sin (C/2).$
9. $\cos^2(A/2) + \cos^2(B/2) - \cos^2(C/2) = 2 + 2 \cos (A/2) \cos (B/2) \sin (C/2).$
10. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$
11. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.$
12. $\cot (A/2) + \cot (B/2) + \cot (C/2) = \cot (A/2) \cot (B/2) \cot (C/2).$
13. $\tan(B/2) \tan (C/2) + \tan (C/2) \tan (A/2) + \tan (A/2) \tan (B/2) = 1.$
14. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$
15. $\sin (A/2) + \sin (B/2) + \sin (C/2)$
$$= 1 + 4 \sin ((\pi - A)/4) \sin ((\pi - B)/4) \sin ((\pi - C)/4)$$
$$= 1 + 4 \cos ((\pi + A)/4) \cos ((\pi + B)/4) \cos ((\pi + C)/4).$$
16. $\cos (A/2) - \cos (B/2) + \cos (C/2)$
$$= 4 \cos ((\pi + A)/4) \cos ((\pi - B)/4) \cos ((\pi + C)/4).$$
17. $\sin (A/2) + \sin (B/2) - \sin (C/2) = -1 + 4 \sin \dfrac{\pi + A}{4} \sin \dfrac{\pi + B}{4} \sin \dfrac{\pi - C}{4}.$
18. $\cos (A/2) + \cos (B/2) + \cos (C/2) = 4 \cos \dfrac{\pi - A}{4} \cos \dfrac{\pi - B}{4} \cos \dfrac{\pi - C}{4}.$
19. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin (A/2) \sin (B/2) \sin (C/2).$

If $A + B + C = 2W$, show that

20. $\sin (W - A) + \sin (W - B) + \sin (W - C) - \sin W$
$$= 4 \sin (A/2) \sin (B/2) \sin (C/2).$$

21. $\cos^2 W + \cos^2(W - A) + \cos^2(W - B) + \cos^2(W - C)$

$$= 2 + 2\cos A \cos B \cos C.$$

If $A + B + C = 0$, then show that

22. $\sin A + \sin B + \sin C = -4\sin(A/2)\sin(B/2)\sin(C/2)$.

23. $\cos^2 A + \cos^2 B - \cos^2 C = 1 + 2\sin A \sin B \cos C$.

24. $\sin 2A + \sin 2B + \sin 2C$

$$= 2(\sin A + \sin B + \sin C) \times (1 + \cos A + \cos B + \cos C).$$

25. If p and q are respectively the product of sines and cosines of the angles of a triangle then the tangets of the angles are the roots of

$$q x^3 - p x^2 + (1 + q) x - p = 0.$$

26. If $A + B + C = 180°$, then

$\cos mA + \cos mB + \cos mC = 1 \pm \sin(mA/2)\sin(mB/2)\sin(mC/2)$, according as m is of the form $4n + 1$ or $4n + 3$; and

$\sin mA + \sin mB + \sin mC = \pm 4 \sin mA \sin mB \sin mC$, according as m is of the form $4n$ or $4n + 2$. Here the sign $\pm$ in the two results can be replaced by $(-1)^{(m-1)/2}$ and $(-1)^{m/2}$ respectively.

6.7 INVERSE CIRCULAR FUNCTIONS

In section 6.4, Example 8 it was observed that the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ was neither one-one nor onto. If we wish to define inverse of the sine function we must cut to size the domain as well as the co-domain (see Remark 2, Section 6.4). Since the range of the sine function is $[-1, 1]$ and it takes all these values once and only once in the interval $[-\pi/2, \pi/2]$, we have the following definition.

Definition. If $f: [-\pi/2, \pi/2] \to [-1, 1]$ is defined by $f(x) = \sin x$ for all x in the domain of f, then f being a bijective function has an inverse f^{-1} which is called the *inverse sine function*. We write $\sin^{-1} x$ (or arc $\sin x$) for $f^{-1}(x)$.

In other words, if $-1 \le x \le 1$, then the numerically smallest angle θ whose sine is x is defined as $\sin^{-1} x$.

EXAMPLE 1. (*i*) $\sin^{-1}(1/\sqrt{2}) = 45°$; (*ii*) $\sin^{-1} 1 = \pi/2$;
(*iii*) $\sin^{-1} 0 = 0$; (*iv*) $\sin^{-1}(-1) = -\pi/2$;
(*v*) $\sin^{-1}(-1/2) = -\pi/6$.

Remark 1. If $0 < x \le 1$ and $\sin^{-1} x = \theta$, then $\sin^{-1}(-x) = -\theta$.

Similarly we proceed to define $\cos^{-1} x$, $\tan^{-1} x$ etc. For the inverse of the cosine function, we observe that the range of the cosine function is again $[-1, 1]$ and it takes all these values exactly once in $[0, \pi]$.

Definition. If $f: [0, \pi] \to [-1, 1]$ is defined by $f(x) = \cos x$, for all x in $[0, \pi]$, then f is a bijective function and its inverse f^{-1} is called the *inverse cosine function*. We write $\cos^{-1} x$ (or arc $\cos x$) for $f^{-1}(x)$.

That is, if $-1 \le x \le 1$, the smallest non-negative angle θ whose cosine is x is defined as $\cos^{-1} x$.

EXAMPLE 2. (*i*) $\cos^{-1}(1/2) = \pi/3$; (*ii*) $\cos^{-1} 1 = 0$;
(*iii*) $\cos^{-1} 0 = \pi/2$; (*iv*) $\cos^{-1}(-1) = \pi$;
(*v*) $\cos^{-1}(-1/2) = 2\pi/3$.

Remark 2. If $0 < x \leq 1$ and $\cos^{-1} x = \theta$, then $\cos^{-1}(-x) = \pi - \theta$.

For the inverse tangent function, we see that the range of the tangent function is the whole real line **R** and it takes every real value just once in $(-\pi/2, \pi/2)$.

Definition. If $f : (-\pi/2, \pi/2) \to \mathbf{R}$ is defined by $f(x) = \tan x$ for all x in $(-\pi/2, \pi/2)$, then f is a bijective function and so has an inverse. The inverse function f^{-1} is called the *inverse tangent function*. Also $f^{-1}(x)$ is denoted by $\tan^{-1} x$ (or arc tan x).

In other words, if x is a real number, then the numerically smallest angle θ whose tangent is x is written $\tan^{-1} x$.

EXAMPLE 3. (*i*) $\tan^{-1}\sqrt{3} = \pi/3$; (*ii*) $\tan^{-1} 0 = 0$

(*iii*) $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$.

Remark 3. If $x > 0$ and $\tan^{-1} x = \theta$, then $\tan^{-1}(-x) = -\theta$.

Definition. The function $f : (0, \pi) \to \mathbf{R}$ defined by $f(x) = \cot x$ for all x in $(0, \pi)$ is a bijective function and its inverse f^{-1} is called the *inverse cotangent function*. We write $\cot^{-1} x$ (or arc cot x) for $f^{-1}(x)$.

EXAMPLE 4. (*i*) $\cot^{-1} 1 = \pi/4$; (*ii*) $\cot^{-1}(-1) = 3\pi/4$;

(*iii*) $\cot^{-1} 0 = \pi/2$; (*iv*) $\cot^{-1}(\sqrt{3} - 2) = 7\pi/12$.

Definition. The function $[0, \pi/2) \cup (\pi/2, \pi] \to \mathbf{R}\backslash(-1, 1)$ given by $f(x) = \sec x$ for all x in the domain of f is a bijective function and its inverse f^{-1} is called the *inverse secant function*. We write $\sec^{-1} x$ (arc sec x) for $f^{-1}(x)$.

EXAMPLE 5. (*i*) $\sec^{-1} 1 = 0$; (*ii*) $\sec^{-1} 2 = \pi/3$;

(*iii*) $\sec^{-1}(-2/\sqrt{3}) = 5\pi/6$; (*iv*) $\sec^{-1}(-1) = \pi$.

Definition. The function $f : [-\pi/2, 0) \cup (0, \pi/2] \to \mathbf{R}\backslash(-1, 1)$ given by $f(x) = \operatorname{cosec} x$ for all x in the domain of f being a bijection has an inverse f^{-1} which is called the *inverse cosecant function*. We write $\operatorname{cosec}^{-1} x$ (or arc cosec x) for $f^{-1}(x)$.

EXAMPLE 6. (*i*) $\operatorname{cosec}^{-1} 1 = \pi/2$; (*ii*) $\operatorname{cosec}^{-1} 2 = \pi/6$;

(*iii*) $\operatorname{cosec}^{-1}(-1) = -\pi/2$; (*iv*) $\operatorname{cosec}^{-1}(-2/\sqrt{3}) = -\pi/3$.

The six functions defined above are called the *inverse circular functions* or *inverse trigonometrical* ratios.

We list some simple properties of these inverse functions.

A. (*a*) $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$, and if $-\pi/2 \leq \theta \leq \pi/2$ then $\sin^{-1}(\sin\theta) = \theta$.

(*b*) $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$, and if $0 \leq \theta \leq \pi$, then $\cos^{-1}(\cos\theta) = \theta$.

(*c*) $\tan(\tan^{-1} x) = x$ for any real x, and if $-\pi/2 < \theta < \pi/2$, then $\tan^{-1}(\tan\theta) = \theta$.

B. (*a*) If $-1 \leq x \leq 1$, then $\sin^{-1} x + \cos^{-1} x = \pi/2$;

(*b*) If x is any real number, then $\tan^{-1} x + \cot^{-1} x = \pi/2$;

(*c*) If $x \geq 1$ or $x \leq -1$, then $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$.

C. (*a*) If $x \geq 1$ or $x \leq -1$, then $\operatorname{cosec}^{-1} x = \sin^{-1}(1/x)$;

(*b*) If $x \geq 1$ or $x \leq -1$, then $\sec^{-1} x = \cos^{-1}(1/x)$;

(*c*) If $x > 0$, then $\cot^{-1} x = \tan^{-1} (1/x)$ and
if $x < 0$, then $\cot^{-1} x = \pi - \tan^{-1} (1/x)$.

D. (*a*) If $-1 \le x \le 1$, then $\sin^{-1} x + \sin^{-1} (-x) = 0$;
(*b*) If $-1 \le x \le 1$, then $\cos^{-1} x + \cos^{-1} (-x) = \pi$;
(*c*) If x is any real number, then $\tan^{-1} x + \tan^{-1} (-x) = 0$.

The graphs of the inverse circular functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are given in Figures 6.23 to 6.28 respectively. The student is advised to note the nuances of these graphs carefully.

EXAMPLE 7. *Show that*

$$\sin^{-1} (3/5) + \sin^{-1} (8/17) = \sin^{-1} (77/85).$$

SOLUTION. Assume that $\sin^{-1} (3/5) = \alpha$, $\sin^{-1} (8/17) = \beta$ and $\sin^{-1} (77/85) = \gamma$, so that $\sin \alpha = 3/5$, $\sin \beta = 8/17$ and $\sin \gamma = 77/85$. Then we have to show that $\alpha + \beta = \gamma$.

Now $\sin (\alpha + \beta)$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$= (3/5)(15/17) + (4/5)(8/17)$$
$$= 77/85 = \sin \gamma.$$

Hence $\alpha + \beta = \gamma$, which proves the result, since we have $\alpha + \beta < \pi/2$.

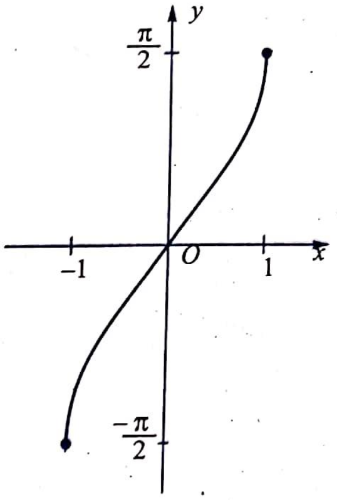


Fig. 6.23

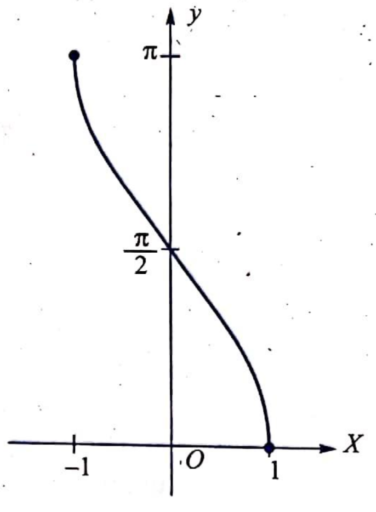


Fig. 6.24

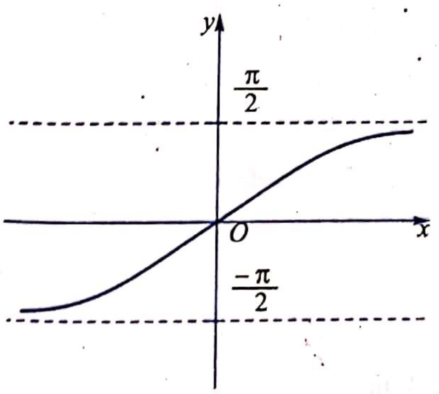


Fig. 6.25

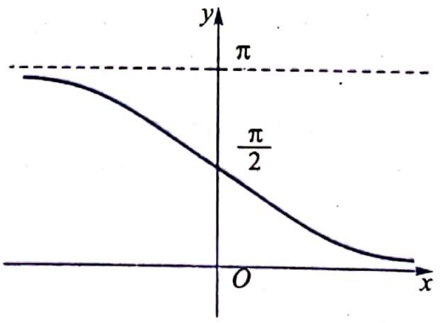


Fig. 6.26

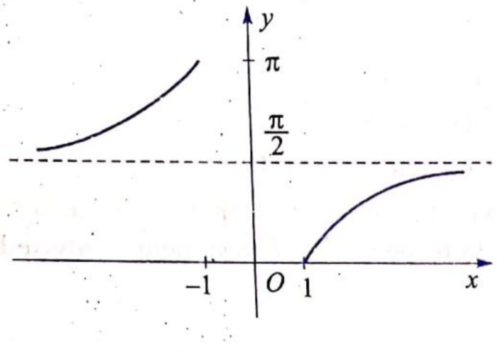


Fig. 6.27

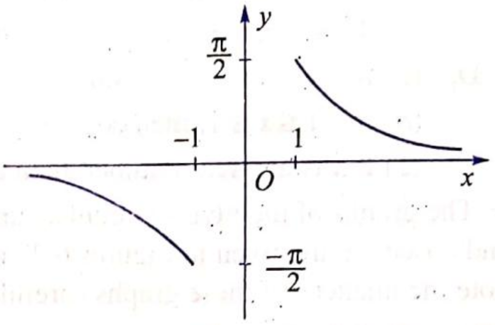


Fig. 6.28

EXAMPLE 8. *Prove that*

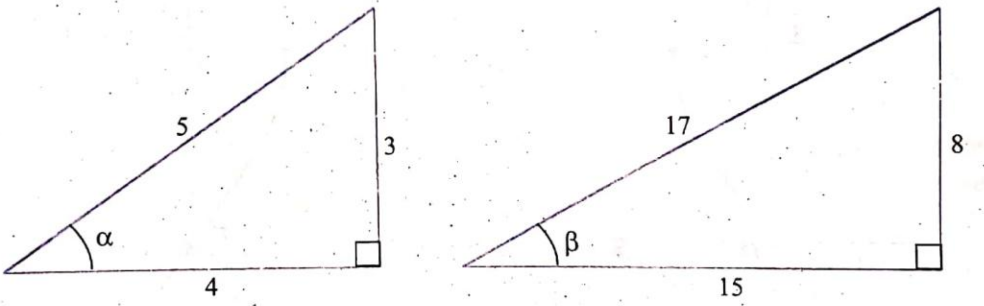
$$4\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4.$$

SOLUTION. Let $\tan^{-1}(1/5) = \alpha$ and $\tan^{-1}(1/239) = \beta$,

so that $\tan\alpha = 1/5$ and $\tan\beta = 1/239$.

We need to show that

$$4\alpha - \beta = \pi/4 \text{ or } 4\alpha = \pi/4 + \beta.$$



(a) (b)

Fig. 6.29

Now $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2(1/5)}{1-(1/5)^2} = 5/12.$$

Therefore, $$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{2(5/12)}{1-(5/12)^2} = 120/119.$$

Also $$\tan\left(\frac{\pi}{4}+\beta\right) = \frac{1+\tan\beta}{1-\tan\beta} = \frac{1+(1/239)}{1-(1/239)} = \frac{120}{119}.$$

Hence $\tan 4\alpha = \tan((\pi/4)+\beta)$, giving $4\alpha = (\pi/4)+\beta$ as desired.

If $ab < 1$, show that

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}.$$

Let $\tan^{-1} a = \alpha$, $\tan^{-1} b = \beta$, $\tan^{-1}\dfrac{a+b}{1-ab} = \gamma$.

Then we have to prove that $\alpha + \beta = \gamma$.

Now $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{a+b}{1-ab} = \tan\gamma$.

Hence $\qquad \alpha + \beta = \gamma$, which is what is wanted.

Question 1. Where have we used the fact that $ab < 1$?

2. What happens if $ab > 1$, that is, how should we change the given relation if $ab > 1$?

EXAMPLE 10. *Solve the equation* $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}(8/31)$.

SOLUTION. Let $\tan^{-1}(x + 1) = \alpha$, $\tan^{-1}(x - 1) = \beta$ so that $\tan\alpha = x + 1$, $\tan\beta = x - 1$. The given relation becomes

$$\alpha + \beta = \tan^{-1}(8/31).$$

This in turn gives $\qquad \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{8}{31}$.

Therefore $\qquad \dfrac{(x+1)+(x-1)}{1-(x+1)(x-1)} = \dfrac{8}{31}$.

Simplifying, we obtain $4x^2 + 31x - 8 = 0$.

Solving this quadratic equation, we have $x = -8, 1/4$.

Here $x = -8$ is inadmissible, because for this value of x, L.H.S. of the given equation is a negative angle, whereas, its R.H.S. is a positive angle. Thus we have only one solution, namely, $x = 1/4$.

EXERCISE 6.7

Prove the following relations: [(1) – (15)].

1. $\cos^{-1}(4/5) + \cos^{-1}(12/13) = \cos^{-1}(33/65)$.
2. $2\cos^{-1}(3/\sqrt{13}) + \cot^{-1}(16/63) + \dfrac{1}{2}\cos^{-1}(7/25) = \pi$.
3. $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.
4. $\tan^{-1}(m/n) + \tan^{-1}((n - m)/(n + m)) = \pi/4$ or $-3\pi/4$, according as $m/n > -1$ or $m/n < -1$.
5. $\tan^{-1}(5/12) + \sin^{-1}(7/25) = \cos^{-1}(253/325)$.
6. $3\tan^{-1}(1/4) + \tan^{-1}(1/20) + \tan^{-1}(1/1985) = \pi/4$.
7. $\tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8) = \pi/4$.
8. $2\tan^{-1}(1/5) + \tan^{-1}(1/7) + 2\tan^{-1}(1/8) = \pi/4$.
9. $\tan^{-1}\dfrac{a-b}{1+ab} + \tan^{-1}\dfrac{b-c}{1+bc} + \tan^{-1}\dfrac{c-a}{1+ca} = 0$.
10. $\cos(2\tan^{-1}(1/7)) = \sin(4\tan^{-1}(1/3))$.
11. $\tan^{-1}\dfrac{a(a+b+c)}{bc} + \tan^{-1}\dfrac{b(a+b+c)}{ca} + \tan^{-1}\dfrac{c(a+b+c)}{ab} = \pi$.
12. $2\tan^{-1}\dfrac{\sqrt{x^2+a^2}-x+b}{\sqrt{a^2-b^2}} + \tan^{-1}\dfrac{x\sqrt{a^2-b^2}}{b\sqrt{x^2+a^2+b^2}} + \tan^{-1}\dfrac{\sqrt{a^2-b^2}}{b} = n\pi$.

13. $\tan^{-1}\dfrac{x\sin\alpha}{1-x\cos\alpha}-\tan^{-1}\dfrac{x-\cos\alpha}{\sin\alpha}=(\pi/2)-\alpha.$

14. $\sin(\cot^{-1}(\cos(\tan^{-1}x)))=\sqrt{\dfrac{x^2+1}{x^2+2}}.$

15. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}=\dfrac{1}{2}\tan^{-1}x.$

16. If $\cos^{-1}(x/a)+\cos^{-1}(y/b)=\alpha$ show that

$$(x^2/a^2)-2(xy/ab)\cos\alpha+(y^2/b^2)=\sin^2\alpha.$$

17. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz$.

18. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi/2$, show that $xy+yz+zx=1$.

19. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, show that $x^2+y^2+z^2+2xyz=1$.

20. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, show that

$$4x^2y^2z^2=2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4.$$

21. If $\theta=\tan^{-1}\dfrac{x\sqrt{3}}{2k-x}$ and $\phi=\tan^{-1}\dfrac{2x-k}{k\sqrt{3}}$,

then show that one of the values of $\theta-\varphi$ is $\pi/6$.

Solve the following equations for x [(22) – (25)].

22. $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\pi/4.$

23. $\cot^{-1}x-\cot^{-1}(x+2)=15°.$

24. $\sin^{-1}x+\sin^{-1}2x=\pi/3.$

25. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x.$

6.8 TRIGONOMETRICAL EQUATIONS

A trigonometrical equation is one which involves one or more circular functions of the unknown angle. In general, the number of solutions is infinite, as the circular functions are periodic. We give below samples of equations which we generally encounter.

1. $\sin\theta=\dfrac{1}{2}.$
2. $\cos^2\theta-\sin\theta=-1.$
3. $\sqrt{3}\cos x-\sin x=1.$
4. $\sin 5x+\sin 3x=2\cos x.$
5. $\tan x+\tan 2x=4/\sqrt{3}.$
6. $\sec x+\operatorname{cosec} x=2\sqrt{2}.$

It is possible that an equation has no solution? For instance the equation $\sin x=2.5$ cannot be solved for x, because the value of $\sin x$ always lies in the interval $[-1, 1]$.

Sometimes we need to know the solutions of equations in a particular range such as $[0, 2\pi]$, $[-\pi/2, \pi/2]$.

EXAMPLE 1. Let us consider the equation $\sin x=(1/2)$. We would like to solve this equation for x. One obvious value of x that strikes our mind is $x=\pi/6$ $(=30°)$. Now we can add (or subtract) an integral multiple of $360°$ $(=2\pi)$ and get more solutions. Thus we have

$$30°, 360° + 30°, 2\,(360°) + 30°, 3\,(360°) + 30°, ...$$
$$-360° + 30°, (-2)\,(360°) + 30°, (-3)\,(360°) + 30°, ...$$

as solutions. All these can be put in the form $n\,360° + 30° = 2n\,\pi + \pi/6$, where n is an integer.

Are there any other solutions of $\sin x = 1/2$? We notice that 1/2 is a positive real number, and the sine function is positive not only in the first quadrant but also in the second quadrant in which the sine function decreases continuously from 1 to 0. The solution in the first quadrant, namely, $x = 30°$ is already taken care of. Since $\sin(180° - \theta) = \sin\theta$, we have $180° - 30° = 150°$ as another solution of $\sin x = 1/2$, Again we generate other solutions by adding integer multiples of 360° to this solution. Thus we obtain 150°, 360° + 150°, 2(360°) + 150°, 3(360°) + 50°, ..., –360° + 150°, (–2) (360°) + 150°, (–3) (360°) + 150°,... as further solutions. These can be abbreviated by the expression

$$n.360° + 150° = 2n\,\pi + 5\pi/6, \text{ where } n \text{ is an integer.}$$

Observe that sine attains the value of 1/2 only once in the first quadrant (at 30°) and only once in the second quadrant (at 150°). Hence the two sets of solutions that have been obtained above contain all the solutions of $\sin x = 1/2$ between them. Thus the solution set S is given by

$$S = \{2n\pi + \pi/6 \mid n \in \mathbf{Z}\} \cup \{2n\pi + 5\pi/6 \mid n \in \mathbf{Z}\}.$$

We can amalgamate, these two sets into one single set as follows:

Observe that $\quad 2n\,\pi + 5\pi/6 = 2n\pi + \pi - \pi/6 = (2n + 1)\,\pi - \pi/6.$

Therefore

$$S = \{2n\pi + \pi/6 \mid n \in \mathbf{Z}\} \cup \{(2n + 1)\pi - \pi/6 \mid n \in \mathbf{Z}\}$$
$$= \{n\pi + (-1)^n\,\pi/6 \mid n \in \mathbf{Z}\},$$

because $n\pi + (-1)^n\,\pi/6$ takes the form $2k\pi + \pi/6$, when n is even (and $n = 2k$) and the form $(2k + 1)\,\pi - p/6$ when n is odd (and $n = 2k + 1$). Also.observe that $\pi/6$ is a solution of the equation with the least magnitude. To summarize, we have that the general solution of the equation $\sin x = 1/2$ is given by $x = n\pi + (-1)^n\,\pi/6$, where n is any integer. Note that $\pi/6$ is a solution which was picked was picked up in the first instance an we manufactured the general solution using this single solution. Similarly one can take the equation $\sin x = -1/2$ and begin with the root $x = -30° = -\pi/6$ and obtain the general solution in the form

$$x = n\pi + (-1)^n\,(-\pi/6),\ n \in \mathbf{Z}.$$

Thus we have the following result.

Theorem 7. Let $x = \alpha$ be one solution of the equation $\sin x = k$, where $-1 \le k \le 1$. Then the general solution is given by $x = n\pi + (-1)^n\,\alpha$, $n \in \mathbf{Z}$.

Proof. 1.2

1.6

We have $\quad \sin x = k = \sin\alpha$; i.e., $\sin x - \sin\alpha = 0.$

Hence, $\quad 2\sin\dfrac{x-\alpha}{2}\cos\dfrac{x+\alpha}{2} = 0$, which gives

$$\sin\frac{x-\alpha}{2} = 0 \text{ or } \cos\frac{x+\alpha}{2} = 0.$$

In the first case, $\frac{x-\alpha}{2} = n\pi, n \in \mathbf{Z}$.

That is, $x = 2n\pi + \alpha, n \in \mathbf{Z}$.

In the second case, $\frac{x+\alpha}{2} = (2n+1)\pi/2, n \in \mathbf{Z}$.

That is, $x = (2n+1)\pi - \alpha, n \in \mathbf{Z}$.

Combining the two solutions we have

$$x = n\pi + (-1)^n \alpha, n \in \mathbf{Z}. \quad \square$$

Remark 1. The one solution that is mentioned in the theorem has to be found either from the knowledge of standard values or from tables.

Remark 2. Note that in the proof we have used the solutions of $\sin x = 0$ and $\cos x = 0$ obtained in an earlier section (see problem 11, Exercise 6.4).

EXAMPLE 2. Now let us look at an equation of the form $\cos x = k, -1 \le k \le 1$. Let us take, for instance, $k = 1/\sqrt{2}$. which gives $\cos x = 1/\sqrt{2}$.

The number $1/\sqrt{2}$ is positive and the cosine function is positive in the first quadrant and the fourth quadrant. The (only) solution in the first quadrant is $x = 45° = \pi/4$ and the solution set that this generates is $\{2n\pi + \pi/4 \mid n \in \mathbf{Z}\}$. The (only) solution in the fourth quadrant is $x = -45° = -\pi/4$. The solution set corresponding to this solution is $\{2n\pi - \pi/4 \mid n \in Z\}$. Hence the general solution is given by $x = 2n\pi \pm \pi/4, n \in Z$. Similarly we can look at equations like $\cos x = -1/\sqrt{2}$. In general, we have the following theorem.

Theorem 8. If $-1 \le k \le 1$ and α is one solution of $\cos x = k$, then the general solution is given by $x = 2n\pi \pm \alpha$, n being any integer.

Proof. We have $\cos x = k = \cos\alpha$; i.e., $\cos x - \cos\alpha = 0$.

Hence $-2\sin\frac{x-\alpha}{2}\sin\frac{x+\alpha}{2} = 0$.

If $\sin\frac{x+\alpha}{2} = 0$, we have $\frac{x+\alpha}{2} = n\pi, n \in \mathbf{Z}$

giving $x = 2n\pi - \alpha, n \in Z$.

If $\sin\frac{x-\alpha}{2} = 0$, we have $\frac{x-\alpha}{2} = n\pi, n \in \mathbf{Z}$

giving $x = 2n\pi + \alpha, n \in \mathbf{Z}$.

Thus the general solution is given by

$$x = 2n\pi \pm \alpha, n \in Z.$$

Remark 3 (a) The solution of $\cos x = 1$ is $x = 2n\pi \pm 0 = 2n\pi, n \in \mathbf{Z}$.

(b) The solution of $\cos x = 0$ is $x = 2n\pi \pm \pi/2, n \in \mathbf{Z}$. That is $x = (4n \pm 1)\pi/2, n \in \mathbf{Z}$. Since numbers of the form $4n + 1, n \in \mathbf{Z}$ and those of the form $4n - 1, n \in \mathbf{Z}$ together exhaust all odd integers, we may write the solutions more compactly in the form $x = (2n + 1)\pi/2, n \in \mathbf{Z}$.

(c) The solution of $\cos x = -1$ is given by $x = 2n\pi \pm \pi, n \in Z$. That is $x = (2n \pm 1)\pi, n \in \mathbf{Z}$. Now numbers of the form $2n + 1, n \in \mathbf{Z}$ are the same as those of the

form $2n - 1$, $n \in \mathbf{Z}$ because both are collections of all odd integers. Hence avoiding duplicity, we can write the solution in the simpler form $x = (2n + 1)\pi$, $n \in \mathbf{Z}$.

Summarizing, we can state that the cosine function (*a*) takes the value 1 at an even multiple of π; (*b*) vanishes at an odd multiple of $\pi/2$; and (*c*) takes the value -1 at an odd multiple of π.

EXAMPLE 3. Now consider an equation of the form $\tan x = k$. Let us take $k = 2 - \sqrt{3}$. Here k is a positive number and the tangent function is known to be positive in the first and third quadrants. The solutions are $x = 15° = \pi/12$ and $x = 180° + 15° = \pi + \pi/12$ respectively in these quadrants. These give rise to the solution sets $\{2n\pi + \pi 12| \, n \in \mathbf{Z}\}$ and $\{2n\pi + \pi + \pi/12 \, | n \in \mathbf{Z}\}$. Their union is $\{2n\pi + \pi/12 \mid n \in \mathbf{Z}\}$ U $\{(2n + 1)\pi + \pi/12| \, n \in \mathbf{Z}\}$ which can be compactly written in the form $\{n\pi + \pi\backslash 12 \mid n \in \mathbf{Z}\}$. We can similarly deal with equations such as $\tan x = -(2 - \sqrt{3})$.

Theorem 9. Let k be any real number and α be a particular solution of the equation $\tan x = k$. Then the general solution is $x = n\pi + \alpha$, where $n \in \mathbf{Z}$.

Proof. We have $\tan x = k = \tan \alpha$. So $\tan x - \tan \alpha = 0$. Simplifying, we have $\sin(x - \alpha) = 0$, and hence $x - \alpha = n\pi$, $n \in Z$, giving $x = n\pi + \alpha$, $n \in Z$. ❑

Remark 4. Equation of the form $\cot x = k$, $\sec x = k$, $\operatorname{cosec} x = k$ can be respectively (and equivalently) transformed into the forms $\tan x = 1/k$ $(k \neq 0)$, $\cos x - 1/k$, $\sin x = 1/k$, which can be solved as above.

EXAMPLE 4. *Solve: $\cos^2 \theta - \sin \theta = -1$.*

SOLUTION. We use the relation $\cos^2 \theta = 1 - \sin^2 \theta$ and rewrite the given equation as a quadratic equation in $\sin \theta$. Accordingly, we have

$$\sin^2 \theta + \sin \theta - 2 = 0.$$

Factorising, we get

$$(\sin \theta - 1)(\sin \theta + 2) = 0.$$

Thus $\sin \theta = 1$ or -2; $\sin \theta = -2$ has no solutions. If $\sin \theta = 1$, we have $\theta = n\pi + (-1)^n \pi/2$, $n \in \mathbf{Z}$, which is the same as $\theta = (4n + 1)\pi/2$, $n \in \mathbf{Z}$.

EXAMPLE 5. *Solve: $\sin 5x + \sin 3x = 2 \cos x$.*

SOLUTION. The given equation can be written as

$$2 \sin 4x \cos x = 2 \cos x.$$

That is, $2 \cos x (\sin 4x - 1) = 0$. So, either $\cos x = 0$ or $\sin 4x = 1$.

The equation $\cos x = 0$ gives $x = (2n + 1)\pi/2$, $n \in Z$.

The equation $\sin 4x = 1$ gives $4x = (4n + 1)\pi/2$, $n \in Z$.

That is, $$x = (4n + 1)\frac{\pi}{8}, \; n \in Z.$$

Hence the solution set is

$$\{(4n + 1)\pi/8 \mid n \in Z\} \cup \{(2n + 1)\pi/2 \mid n \in Z\}.$$

Remark 5. Suppose we are asked to solve the equation in Example 5 in $[0, 2\pi]$. Then $\cos x = 0$ has two solutions, $x = \pi/2$ and $x = 3\pi/2$, while $\sin 4x = 1$ has four solutions, $x = \pi/8$, $x = 5\pi/8$, $x = 9\pi/8$, $x = 13\pi/8$. In all, we have six solutions.

EXAMPLE 6. *Solve the equation*

$$a \cos x + b \sin x = c.$$

What is the condition to be satisfied by a, b, c for the existence of a solution? Hence solve $\sqrt{3}\ \cos x - \sin x = 1$.

We know that for any ordered pair (a, b) of real numebrs (not simultaneously zero), there exists a corresponding pair of real numbers r and θ such that $r > 0$;

$$a = r \cos \theta \text{ and } b = r \sin \theta.$$

To see this, plot the point $P = (a, b)$ in the xy – plane (P may lie in any quadrant or an any axis). Join OP. Let $OP = r$ and $\angle xOP = \theta$, where θ is measured in the positive direction. From the very definitions of sin θ and cos θ, we have $\cos \theta = a/r$ and $\sin \theta = b/r$, where $a = r \cos \theta$ and $b = r \sin \theta$.

Here $$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(b/a) \text{ or } \pi + \tan^{-1}(b/a).$$

The angle θ should satisfy $r \cos \theta = a$ and $r \sin \theta = b$ and so lies in the same quadrant as $P = (a, b)$.

The given equation then becomes $r \cos \theta \cos x + r \sin \theta \sin x = c$.

Hence $r \cos (x - \theta) = c$; *i.e.*, $\cos (x - \theta) = c/r$.

For this equation to have a solution, we must have

$$-1 \le c/r \le 1,$$

which is the same as

$$-\sqrt{a^2 + b^2} \le c \le \sqrt{a^2 + b^2}$$

If $\cos \alpha = c/r$, then the general solution is given by

$$x - \theta = 2n\pi \pm \alpha,\ n \in Z.$$

giving

$$x = 2n\pi + \theta \pm \alpha,\ n \in Z.$$

Thus a criterion for the solvability of the equation is

$$|c| \le \sqrt{a^2 + b^2}\ .$$

In simple cases, it is advisable to divide the given equation directly by $\sqrt{a^2 + b^2}$ and recognise $\dfrac{a}{a^2 + b^2}$ and $\dfrac{b}{a^2 + b^2}$ as cos θ and sin θ for some suitable angle θ. We apply this method to the given numerical problem. Dividing the equation by $\sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, we obtain $(\sqrt{3}/2) \cos x - (1/2) \sin x = 1/2$.

Observing that $\sqrt{3}/2 = \cos \pi/6$ and $1/2 = \sin \pi/6$, we get $\cos \pi/6 \cos x - \sin \pi/6 \sin x = 1/2$. That is, $\cos (x + \pi/6) = 1/2 = \cos \pi/3$, which gives $x + \pi/6 = 2n\pi \pm \pi/3$, $n \in Z$. Hence the solution is $x = 2n\pi - \pi/6 \pm \pi/3$, $n \in Z$.

EXAMPLE 7. *Solve the equation*

$$\sin (x + \pi/4) = \sin 2x.$$

SOLUTION. At the first glance, we may be tempted to use the formula for $\sin C - \sin D$ which anyway gives the correct solution. But we may also argue in the following manner:

The value $2x$ may be considered as a particular solution of $x + \pi/4$ and hence the general solution is given by

$$x + \pi/4 = n\pi + (-1)^n\, 2x,\ n \in Z,$$

since the sine relation is involved.

Solving this for x, we get

$$x = \frac{n\pi - \pi/4}{1-(-1)^n 2},\ n \in Z.$$

Compare this with the solution obtained by the first method, namely,

$$x = \frac{\pi}{4} + 2n\,\pi,\ n \in Z \text{ or } \frac{(2n+1)\pi - \pi/4}{3},\ n \in Z.$$

In fact these two solution sets are the same !

EXAMPLE 8. *Solve: sec x + cosec x =* $2\sqrt{2}$.

SOLUTION. The given equation is the same as

$$\sin x + \cos x = 2\sqrt{2}\ \sin x \cos x.$$

Dividing by $\sqrt{2}$ and using the fact that $\sin \pi/4 = 1/\sqrt{2} = \cos \pi/4$, we get

$$\sin (x + \pi/4) = \sin 2x,$$

which is the same as the equation in Example 7.

EXERCISE 6.8

Solve the equations (1) – (12) for θ.

1. $2 \cos 2\theta - 7 \cos \theta = 0.$
2. $\sin 3\theta + 5 \sin \theta = 0.$
3. $\sin (m + n)\,\theta + \sin (m - n)\,\theta = \sin m\theta.$
4. $\tan 5\theta + \cot 2\theta = 0.$
5. $\tan \theta + \tan 2\theta + \tan 3\theta = 0.$
6. $\cos 3\theta = \cos^3 \theta.$
7. $(\sqrt{2} - 1) \cos \theta + \sin \theta = 1.$
8. $2 \cos \theta + 3 \sin \theta = 3.$
9. $\sqrt{3} \sin \theta + \cos \theta = \cos (\pi/5) \sec \theta.$
10. $\tan \theta + \tan (\theta + \pi/3) + \tan (\theta - \pi/3) = 3.$
11. $\cot \theta + \cot 2\theta + \cot 3\theta = 0.$
12. $\sin \theta + \sin 2\theta + \sin 3\theta = 0.$
13. Explain how to solve $a \cos^2 \theta + 2b \cos \theta \sin \theta + c \sin^2 \theta = d$. What is the condition governing a, b, c, d for the existence of a solution?
14. Solve the equation $a \cos \theta + b \sin \theta = c$, by transforming this into a quadratic equation in $\tan \theta/2$. Hence solve $(\sqrt{2} - 1) \cos \theta + \sin \theta = 1.$
15. Solve the equation $\cos \theta + \sin 2\theta = 0$ by the following two methods:
    (*a*) Write the equation as $\cos \theta\, (1 + 2 \sin \theta) = 0$ and solve $\cos \theta = 0$, $2 \sin \theta + 1 = 0$ separately.

(*b*) Write the equation as $\sin 2\theta = \sin(\theta - \pi/2)$ and solve as in. Example 7. Compare the two solution sets obtained in (*a*) and (*b*) and show that they are the same.

16. Solve the equation for θ:

$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, A, B, C being the angles of a triangle.

17. Find the fallacy in the following argument: Let us solve the equation

$$2\cos^2\theta = \sin\theta + \sin 3\theta - 5\cos\theta.$$

This is the same as

$$2\cos^2\theta = 2\sin 2\theta\cos\theta - 5\cos\theta.$$

Hence, $2\cos\theta = 2\sin 2\theta - 5.$

That is, $2\sin 2\theta - 2\cos\theta = 5.$

Now $|2\sin 2\theta - 2\cos\theta| \le 2 + 2 = 4$ and so the given equation has no solution. But $\theta = \pi/2$ is a solution of the given equation, as can easily be checked.

18. Prove the following relations:

(*a*) $\{2n\pi + \alpha \mid n \in \mathbf{Z}\} \cup \{(2n+1)\pi - \alpha \mid n \in \mathbf{Z}\}$
$= \{n\pi + (-1)^n \alpha \mid n \in \mathbf{Z}\}.$

(*b*) $\{2n+1 \mid n \in \mathbf{Z}\} = (2n-1 \mid n \in \mathbf{Z}\}.$

(*c*) $\{4n+1 \mid n \in \mathbf{Z}\} \cup \{4n-1 \mid n \in \mathbf{Z}\} = \{2n+1 \mid n \in \mathbf{Z}\}.$

(*d*) $\left\{\dfrac{n\pi - \pi/4}{1-(-1)^n 2} \middle| n \in \mathbf{Z}\right\} = \{\pi/4 - 2n\pi \mid n \in Z\} \cup \left\{\dfrac{(2n+1)\pi - \pi/4}{3} \middle| n \in \mathbf{Z}\right\}$

(*see Example 7*).

19. Find the minimum and maximum values of the following expressions in θ:

(*a*) $\cos\theta + \sin\theta.$

(*b*) $\sqrt{3}\cos\theta - \sin\theta.$

(*c*) $3\sin\theta + 4\cos\theta + 2.$

(*d*) $a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta.$

(*e*) $\cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha}).$

20. Solve the system for x and y:

$$\tan x + \tan y = (4/3)\sqrt{3}; \quad \tan 2x + \tan 2y = 0.$$

21. Draw the graphs of the following equations:

(*a*) $y = \cos x + \sin x$

(*b*) $y = \sqrt{3}\cos x - \sin x + 1.$

22. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then show that

$$\theta = (1/4)\,[(2n+1)\pi \pm \cos^{-1}(1/8)],\ n \in Z.$$

6.9 PROPERTIES OF TRIANGLES

The triangle is one of the simplest geometrical figures and has many interesting properties. Associated with a triangle are some special points, circles and distances (lengths). We study the properties of these points and circles in this section. If ABC is a triangle, the six quantities namely the three angles A, B, C and the three sides $BC = a$, $CA = b$, $AB = c$ are called the *elements* of the triangle.

Also the semiperimeter $\dfrac{a+b+c}{2}$ of triangle ABC is denoted by s.

A. The circumcentre and the sine and cosine rules.

Theorem 10. (*The Sine Rule*) : In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof.

Case I. Suppose A is an acute angle in the acute-angled triangle ABC. Then the circumcentre S is in the interior of the triangle. Join BS and produce it to meet the circumcircle in D. Join DC. Then as A and D are on the same side of BC, we have $\angle BDC = \angle BAC = A$. But from the right triangle BDC in which $\angle BCD = 90°$, we have $\sin \angle BDC = BC/BD$.

That is, $\quad \sin A = \dfrac{a}{2R}$. So $\dfrac{a}{\sin A} = 2R.$

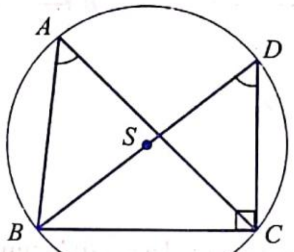
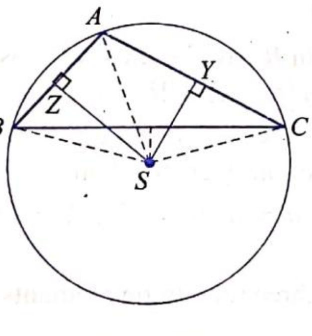


Fig. 6.30

Case II. Suppose A is an obtuse angle. Then the circumcentre S is outside the triangle



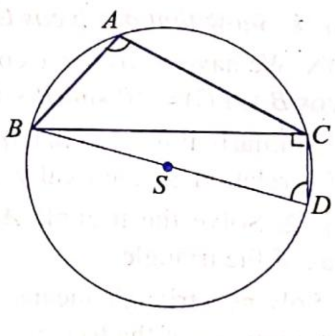


Fig. 6.31 **Fig. 6.32**

ABC and in fact A and S lie on the opposite sides of BC (Fig. 6.31, 6.32). Produce BS to meet the circumcircle in D and join CD. Then as A and D are on the opposite sides of BC, we have $\angle BDC = 180° - \angle BAC = 180° - A$. Also from triangle BDC which is right-angled at C, we get $\sin \angle BDC = BC/BD$.

That is, $\quad \sin(180° - A) = \dfrac{a}{2R}.$

But $\quad \sin(180° - A) = \sin A$. Hence $\sin A = \dfrac{a}{2R}$ and $\dfrac{a}{\sin A} = 2R.$

Case III. Let A be a right angle. Then S is the midpoint of the hypotenuse BC (Fig. 6.33, 6.34), which is a diameter of the circumcircle. so $a = BC = 2R$ and

$$\frac{a}{\sin A} = \frac{2R}{\sin 90^\circ} = \frac{2R}{1} = 2R.$$

Thus we have proved that $a/(\sin A) = 2R$ in all cases. Similarly one proves that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

This is the Sine Rule (or the Sine Law). ❑

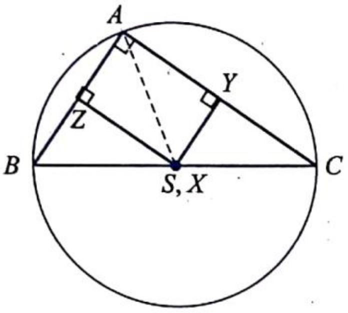


Fig. 6.33

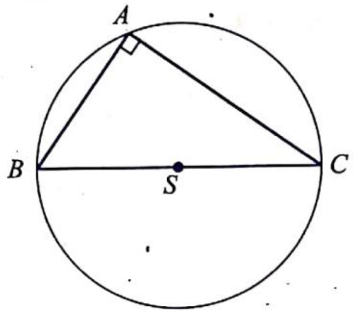


Fig. 6.34

Remark 1. The reader might have noticed a small lacuna in the proof in Case I. Though A is an acute angle, the triangle itself need not be acute-angled. The acute angle A may lie in a right-angled triangle or obtuse-angled triangle. These two cases may be dealt with similarly and the reader should convince himself of the validity of the sine rule.

From the above theorem, we have $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$.

EXAMPLE 1. *Show that $a = b \cos C + c \cos B$.*

SOLUTION. We have $b \cos C + c \cos B = 2R \sin B \cos C + 2R \sin C \cos B = 2R (\sin B \cos C + \cos B \sin C) = 2R \sin (B + C) = 2R \sin (\pi - A) = 2R \sin A = a$.

Remark 2. Similarly it follows that $b = c \cos A + a \cos C$, $C = a \cos B + b \cos A$. One can prove this relation geometrically using projections [see problem 13, Exercise 6.9].

EXAMPLE 2. Solve the triangle ABC, given $a = 6$, $B = 45°$, $A = 75°$. Find the circumradius of the triangle.

Remark 3. Solving a triangle means to find the three remaining elements, given three independent elements of the triangle.

SOLUTION. We have $C = 180° - (A + B) = 60°$.

From the Sine Rule, $b = \dfrac{a}{\sin A} \sin B$

$$= \frac{6}{\sin 75^\circ} \sin 45^\circ = \frac{6.}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} \cdot \frac{1}{\sqrt{2}} = \frac{12}{\sqrt{3}+1} = 6(\sqrt{3}-1).$$

Again from the same rule.

$$c = \frac{a}{\sin A} \sin C = \frac{6}{\sin 75^\circ} \sin 60^\circ$$

$$= \frac{6}{\frac{\sqrt{3}+1}{2\sqrt{2}}} \frac{\sqrt{3}}{2} = \frac{6\sqrt{6}}{\sqrt{3}+1} = 3\sqrt{6}(\sqrt{3}-1).$$

$$R = \frac{a}{2 \sin A} = \frac{6}{2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \frac{6\sqrt{2}}{\sqrt{3}+1} = 3(\sqrt{6}-\sqrt{2}).$$

Theorem 11. (The *Cosine Rule*) In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$
$$b^2 = c^2 + a^2 - 2ca \cos B,$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Proof.

Case I. Let A be an acute angle in acute-angled triangle ABC. Draw CD perpendicular to AB. The point D falls within the interior of side AB (Fig. 6.35). We have

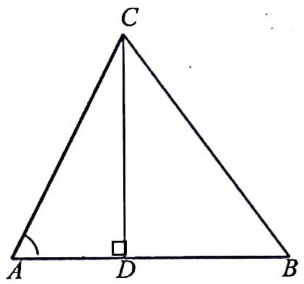


Fig. 6.35

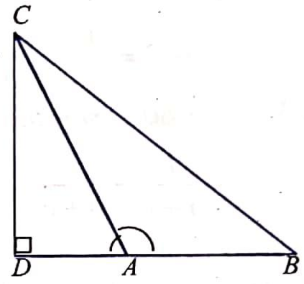


Fig. 6.36

$$BC^2 = CD^2 + DB^2 \quad \text{(from the right triangle } BDC\text{)}$$
$$= CD^2 + (AB - AD)^2 = CD^2 + AB^2 + AD^2 - 2AB{\cdot}AD$$
$$= (AD^2 + CD^2) + AB^2 - 2AB{\cdot}AD$$
$$= AC^2 + AB^2 - 2AB{\cdot}AD, \quad \text{(from the right triangle } ACD\text{).}$$

Now from triangle ACD, $\cos \angle CAD = AD/DC$.

That is, $AD = AC \cos A$.

So $BC^2 = AC^2 + AB^2 - 2AB{\cdot}AC \cos A$.

That is, $a^2 = b^2 + c^2 - 2\,bc \cos A$.

Case II. Let A be an obtuse angle. Draw CD perpendicular to BA extended. The point D falls outside the line segment AB (Fig. 6.36).

We have
$$BC^2 = CD^2 + DB^2 \quad \text{(from the right triangle } BCD\text{)}$$
$$= CD^2 + (DA + AB)^2$$
$$= CD^2 + DA^2 + AB^2 + 2DA{\cdot}AB$$
$$= AC^2 + AB^2 + 2DA{\cdot}AB \quad \text{(from the right triangle } ACD\text{)}$$

From triangle ACD, $\cos \angle DAC = AD/AC$.

So $AD = AC \cos (180° - A) = -AC \cos A$.

Hence $BC^2 = AC^2 + AB^2 - 2AB{\cdot}AC \cos A$.

That is, $a^2 = b^2 + c^2 - 2bc \cos A$.

Case III. Let A be a right angle (Fig. 6.37). Straightaway

$$b^2 + c^2 - 2bc \cos A = b^2 + c^2 - 2bc \cos 90°$$
$$= b^2 + c^2 = a^2.$$

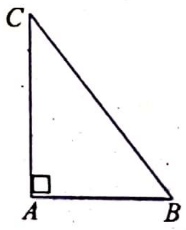


Fig. 6.37

Thus in all cases,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly the other two relations can be proved.

This is the Cosine Rule (or the Cosine Law). ❑

Remark 4. As before the proof in case *I* is incomplete. The acute angle A may also lie in a right triangle or an obtuse triangle *ABC*. The cosine rule may be proved in these two cases also similarly.

From the cosine rule, one has

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

EXAMPLE 3. *If C = 60°, show that*

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

Prove the converse also.

SOLUTION. The given relation holds good

iff $(a + b + 2c)(a + b + c) = 3(a + c)(b + c)$,

i.e., iff $a^2 + b^2 - ab = c^2$,

i.e., iff $a^2 + b^2 - ab = a^2 + b^2 - 2ab \cos C$,

i.e., iff $\cos C = 1/2$,

i.e., iff $C = 60°$.

Note. The Exercises under this Section 6.9 A would form part of Exercise 6.9 B to come after the next Section.

B. Ratios of A/2 and the area of a triangle:

From the identity $\cos A = 1 - 2 \sin^2 (A/2)$, we have

$$\sin^2 (A/2) = 1 - \cos A$$

$$= 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}.$$

Since $2s = a + b + c$, we have $a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c)$

and $a - b + c = 2(s - b)$, similarly.

So $$2 \sin^2(A/2) = \frac{4(s-b)(s-c)}{2bc}.$$

Therefore $$\sin^2(A/2) = \frac{(s-b)(s-c)}{bc}.$$

Taking square roots on both sides and observing that sin (*A*/2) is always positive (because $0 < A/2 < 90°$), we get

$$\sin(A/2) = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Using the identity $\cos A = 2\cos^2(A/2) - 1$, one proves similarly that

$$\cos(A/2) = \sqrt{\frac{s(s-a)}{bc}}.$$

So $$\tan(A/2) = \frac{\sin(A/2)}{\cos(A/2)} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ and } \cot(A/2)$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

The ratios of $B/2$ and $C/2$ are similarly obtained.

Again $$\sin A = 2\sin(A/2)\cos(A/2) = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The expressions for $\sin B$ and $\sin C$ are similarly written down.

Theorem 12. (Area of a triangle). If Δ denotes the area of triangle ABC, then

$$\Delta = (1/2)bc\sin A = (1/2)ca\sin B = (1/2)ab\sin C.$$

The proof is left to the reader. One has only to use the formula area = (1/2) (base) (height) for a triangle and consider the three cases A acute, A obtuse and A a right angle. The case A acute has three subcases as usual. ❑

Corollary 1. $$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \textit{(Heron's formula)}$$

$$= 2R^2 \sin A \sin B \sin C.$$

Proof. We have $$\Delta = (1/2)\,bc\sin A$$

$$= (1/2)\,bc \cdot \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Also $$\Delta = (1/2)\,bc\sin A$$

$$= (1/2)\cdot 2R\sin B \cdot 2R\sin C \cdot \sin A$$

$$= 2R^2 \sin A \cdot \sin B \cdot \sin C.$$ ❑

Corollary 2. (*a*) $\Delta = \dfrac{abc}{4R}$; (*b*) $R = \dfrac{abc}{4\Delta}$.

The proofs of these simple relations are left to the reader. ❑

EXAMPLE 4. *If in triangle ABC, $a = 13$, $b = 4$, $c = 15$ find Δ and R.*

SOLUTION. We have $s = \dfrac{a+b+c}{2} = 16$; so $s-a = 16-13 = 3$; $s-b = 16-4 = 12$; $s-c = 16-15 = 1$.

Hence $$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16\times 3\times 12\times 1} = 24 \text{ square units.}$$

$$R = \frac{abc}{4\Delta} = \frac{13\times 4\times 15}{4\times 24} = 8\frac{1}{8} \text{ units.}$$

EXAMPLE 5. *Show that, in a triangle ABC,*

$$\frac{\cot A + \cot B + \cot C}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{a^2 + b^2 + c^2}{(a+b+c)^2}.$$

SOLUTION. We have $\cot A = \dfrac{\cos A}{\sin A} = \dfrac{b^2 + c^2 - a^2}{2bc \sin A} = \dfrac{b^2 + c^2 - a^2}{4\Delta}$

Similar expressions for $\cot B$ and $\cot C$ can be written down.

So $\cot A + \cot B + \cot C$

$$= \frac{b^2 + c^2 - a^2}{4\Delta} + \frac{c^2 + a^2 - b^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta}$$

$$= \frac{a^2 + b^2 + c^2}{4\Delta}$$

Again $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2}$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{\sqrt{s}\sqrt{s}[3s-(a+b+c)]}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{s(3s-2s)}{\Delta} = \frac{s^2}{\Delta}.$$

Hence L.H.S. of the given relation

$$= \frac{a^2 + b^2 + c^2}{4\Delta} \div \frac{s^2}{\Delta} = \frac{a^2 + b^2 + c^2}{4s^2} = \text{R.H.S.}$$

EXERCISE 6.9 B

1. Show that (*a*) $\dfrac{a+b}{c} = \dfrac{\cos \frac{A-B}{2}}{\cos \frac{A+B}{2}}$;

(*b*) $\dfrac{a-b}{a+b} = \dfrac{\sin \frac{A-B}{2}}{\sin \frac{A+B}{2}}$; (*c*) $\dfrac{b-c}{b+c} \cot \dfrac{A}{2} = \tan \dfrac{B-C}{2}$.

2. Prove *Appollonius's Theorem* using the Cosine Rule : 'If ABC is a triangle and D is the midpoint of BC, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.' Hence evaluate the lengths of the medians of a triangle in terms of its sides. (You will find the answers in Theorem 28 of Chapter 4).

3. Prove the following generalisation of the result in Problem 2 above : 'If ABC is a triangle and D is a point on BC dividing it internally in the ratio $m : n$, then

$$(m+n)\cdot AD^2 = m\cdot AC^2 + n\cdot AB^2 - \frac{mn}{m+n}\cdot BC^2.$$

What is the corresponding result if D divides BC externally in the ratio $m : n$?

4. If the internal bisector of angle A meets the opposite side BC in D, show that

$$AD = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

Hence or otherwise show that if the internal bisectors of two angles are equal in a triangle, then the triangle is isosceles.

5. If in a triangle ABC, $a = 13$, $b = 4$, $c = 15$, find its altitudes h_a, h_b, h_c.

6. If in a triangle ABC, h_a, h_b, h_c are its altitudes and $a \geq b \geq c$., show that $a + h_a \geq b + h_b \geq c + h_c$.

7. If SX, SY, SZ are the perpendiculars dropped on the sides BC, CA, AB of a triangle ABC from its circumcentre S, show that $SX = R\cos A$, $SY = R\cos B$, $SZ = R\cos C$.

8. Prove that, in any triangle ABC,

   (i) $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$;

   (ii) $a^3\sin(B-C) + b^3\sin(C-A) + c^3\sin(A-B) = 0$.

9. Show that, in a triangle ABC,

   (i) $4\Delta = b^2\sin 2C + c^2\sin 2B$;

   (ii) $\Delta = \dfrac{a^2}{2(\cot B + \cot C)}$;

   (iii) $16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$.

10. If P is an interior point in a triangle ABC, then

    $\sin\angle PAB\cdot\sin\angle PBC\cdot\sin\angle PCA = \sin\angle PBA\cdot\sin\angle PCB\cdot\sin\angle PAC$.

    State and prove the converse of this result.

11. If D is a point on the side BC of a triangle ABC such that $BD : DC = m : n$, and $\angle ADC = \theta$, $\angle DAB = \alpha$, $\angle DAC = \beta$, then show that

    (i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$,

    (ii) $(m+n)\cot\theta = n\cot B - m\cot C$.

12. If P is an interior point of a triangle ABC, such that $\angle PAB = \angle PBC = \angle PCA = w$, then show that

    (i) $\cot w = \cot A + \cot B + \cot C$;

    (ii) $\operatorname{cosec}^2 w = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

13. (a) Prove geometrically that $a = (b\cos C) + (c\cos B)$.

    (b) Prove the cosine rule using the sine rule [Hint : First prove $a = b\cos C + c\cos B$ and two similar relations (see Example 1). Then solve for $\cos A$, $\cos B$, $\cos C$].

14. Given the numerical values of sides a, b, c of a triangle ABC, its angles A, B, C can be found by using any of the expressions for the sines, the cosines of the angles A, B, C, or the sines, the cosines, the tangents, of the angles $A/2$, $B/2$, $C/2$. But generally the expressions $\tan A/2$, $\tan B/2$, $\tan C/2$ are preferred to the others while using the logarithmic tables. Explain why.

15. Two triangles have the same perimeter and area. If the sides of one triangle are 51, 35, 26 and one side of the other triangle is 41, find the remaining sides of the latter.

16. Solve the triangle ABC, given side a, angle A and the product $k = b\,c$ of the other two sides. Under what conditions does the triangls exist?

17. Let an n-sided regular polygon of side a inscribed in a circle of radius R and circumscribed about a circle of radius r. Find the area of the polygon in terms of (*i*) a (*ii*) r (*iii*) R. Also express R and r in terms of a.

18. Suppose a triangle ABC is to be solved given b, c and B. Use the sine rule to show that (*i*) if B is acute; then no triangle, one right triangle, two triangles or one triangle exists according as $b < c \sin B$, $b = c \sin B$, $c > b > c \sin B$ or $b > c$. (*ii*) if B is greater than or equal to 90°, then exactly one solution exists if $b > c$ and none otherwise.

19. Discuss the ambiguous case of Question 21 above algebraically considering the relation $b^2 = c^2 + a^2 - 2\,c\,a \cos B$ as a quadratic equation in a and solving it for a.

20. Solve the following triangles:

(*i*) $a = 121, b = 94, c = 135$.
(*ii*) $b = 108, c = 77, A = 76°$.
(*iii*) $a = 55, A = 65°, B = 38°$.
(*iv*) $b = 37, c = 42, B = 52°$.
(*v*) $a = 125, b = 112, A = 40°$.
(*vi*) $c = 13.1, a = 17.5, C = 49°$.
(*vii*) $b = 1, c = \sqrt{6} - \sqrt{2}, B = 75°$.
(*viii*) $a = 100, b = 80, A = 130°$.
(*ix*) $a = 70, c = 56.3, C = 100°$.
(*x*) $a = 3, b = 5, c = 7$.

C. The Incentre and the Ex-centres:

From Theorem 24 of Chapter 3, we know that the internal bisectors of the angles of a triangle are concurrent and that the point of concurrence is equidistant from the sides. We recall that r denotes the inradius of a triangle.

Theorem 13. In a triangle ABC, the inradius is given by

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

$$= 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Proof. (*i*) With reference to Figure 6.38

$$\Delta ABC = \Delta IBC + \Delta ICA + \Delta IAB$$

That is,

$$\Delta = (1/2)BC \cdot ID + (1/2)CA \cdot IE + (1/2)AB \cdot IF$$
$$= (1/2)\,(ar + br + cr)$$
$$= (1/2)(a + b + c)\,r = sr$$

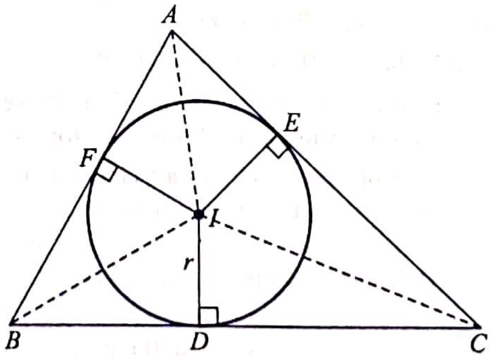


Fig. 6.38

Hence $r = \dfrac{\Delta}{s}$ (as already seen in Theorem 37 of Chapter 4).

(*ii*) Again from triangles IBD and ICD, we have

$$BD = ID \cot \angle IBD = r \cot(B/2),$$

and $$CD = ID \cot \angle ICD = r \cot(C/2).$$

Adding we have

$$a = r(\cot B/2 + \cot C/2)$$

So
$$r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{2R \sin A \sin(B/2)\sin(C/2)}{\sin\frac{B+C}{2}}$$
$$= (2R \cdot 2\sin(A/2)\cos(A/2)\sin(B/2)\sin(C/2))/\cos(A/2)$$
$$= 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

(*iii*) Since AE and AF are tangents to the incircle we have $AE = AF$.

Similarly $EF = BD$, $CD = CF$.

Now
$$2s = a + b + c = a + AE + EC + AF + FB$$
$$= a + 2AE + CD + BD = a + 2AE + BC$$
$$= 2(a + AE).$$

So $AE = s - a = AF$ (as already seen in Theorem 33 of Chapter 4).

So $r = (s-a)\tan\frac{A}{2}$.

Similarly from triangles IBD and ICD, we obtain

$$r = (s-b)\tan\frac{B}{2} \text{ and } r = (s-c)\tan\frac{C}{2}.$$ ❑

Again from Theorem 30 of Chapter 4, we know that the external bisectors of any two angles of a triangle and the internal bisector of the third angle are concurrent and that the point of concurrence is equidistant from all the three sides of the triangle. Let r_1, r_2, r_3 denote the ex-radii of the triangle. Note that in chapter 4 we have called these r_a, r_b, r_c.

Theorem 14. In a triangle ABC,

$$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Proof. (*i*) With reference to Figure 6.39,

quadrilateral
$$ABI_1C = \Delta ABI_1 + \Delta ACI_1$$
$$= \Delta ABC + \Delta I_1BC$$

Hence
$$\Delta ABC = \Delta I_1CA + \Delta I_1AB - \Delta I_1BC$$

That is,
$$\Delta = (1/2)AC.I_1F_1 + (1/2)AB.I_1F_1 - (1/2)BC.I_1D_1$$
$$= (1/2)(br_1 + cr_1 - ar_1)$$
$$= (1/2)(b + c - a)r_1 = (s-a)r_1$$

Therefore
$$r_1 = \frac{\Delta}{s-a}.$$

Similarly
$$r_2 = \frac{\Delta}{s-b} \text{ and } r_3 = \frac{\Delta}{s-c}.$$

Recall that this is Theorem 37(2) of Chapter 4.

(*ii*) Again from triangles $I_1\ BD$ and $I_1\ CD$,

$$BD_1 = I_1D_1 \cot \angle I_1BD$$
$$= r_1 \cot(90° - B/2)$$
$$= r_1 \tan(B/2);$$

and $$CD_1 = I_1D_1 \cot \angle I_1CD_1$$
$$= r_1 \tan(C/2).$$

Adding we get $a = r_1(\tan B/2) + \tan(C/2))$

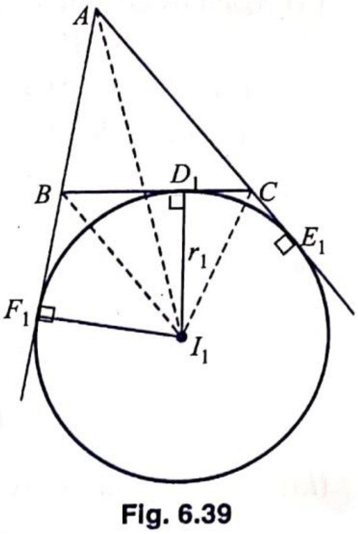


Fig. 6.39

So $$r_1 = \frac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}}$$

$$= \frac{2R \sin A \cdot \cos(B/2) \cos(C/2)}{\sin\frac{(B+C)}{2}}$$

$$= 2R \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \div \cos\frac{A}{2}.$$

$$= 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(Similarly $r_2 = 4R \sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}$

and $r_3 = 4R \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$)

(*iii*) Finally, since AE_1 and AF_1 are tangents to the excircle, we have $AE_1 = AF_1$. Similarly $BD_1 = BF_1$ and $CD_1 = CE_1$.

So $$2s = a + b + c = BC + CA + AB$$
$$= BD_1 + D_1C + CA + AB = (AB + BF_1) + (AC + CE_1)$$
$$= AF_1 + AE_1 = 2AE_1$$

That is, $AE_1 = s = AF_1$

[Note further that $BD_1 = BF_1 = AF_1 - AB = s - c$, and $CD_1 = CE_1 = AE_1 - AC = s - b$ Recall Theorem 34 and its corollary, from Chapter 4].

Hence from triangle I_1AE_1, we have

$$r_1 = I_1E_1 = AE_1 \tan IAE_1 = s \tan\frac{A}{2}.$$

$$\left(\text{Similarly } r_2 = s\tan\frac{B}{2} \text{ and } r_3 = s\tan\frac{C}{2}.\right)$$ □

EXAMPLE 6. *Show that* $r_1 - r_2 + r_3 + r = 4R \cos B$.

SOLUTION. We have $r_1 - r_2 + r_3 + r = (r_1 + r_3) - (r_2 - r)$.

Now
$$r_1 + r_3 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} + 4R \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}$$
$$= 4R \cos\frac{B}{2} \sin\frac{A+C}{2} = 4R \cos^2\frac{B}{2}.$$

Again,
$$r_2 - r = 4R \cos\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2} - 4R \sin\frac{B}{2} \sin\frac{A}{2} \sin\frac{C}{2}$$
$$= 4R \sin\frac{A}{2} \cos\frac{A+C}{2} = 4R \sin^2\frac{B}{2}.$$

Hence
$$r_1 - r_2 + r_3 + r = 4R \cos^2\frac{B}{2} - 4R \sin^2\frac{B}{2}$$
$$= 4R\left(\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\right) = 4R \cos B.$$

EXAMPLE 7. *If one of the ex-radii of a triangle is equal to its semiperimeter, then the triangle is right-angled.*

SOLUTION. Let $r_1 = s$, We know that $r_1 = s \tan\frac{A}{2}$.

Hence $\tan\frac{A}{2} = 1$ which means $\frac{A}{2} = 45°$. That is, $A = 90°$.

EXAMPLE 8. *If* $r : b + c : a = 2 : 17 : 13$, *then the triangle is right-angled.*

SOLUTION. Let $r = 2\lambda,\ b + c = 17\lambda,\ a = 13\lambda$.

We have
$$r = (s - a)\tan\frac{A}{2} = \frac{1}{2}(b + c - a)\tan\frac{A}{2}$$

$$\therefore \quad \tan\frac{A}{2} = \frac{2r}{b+c-a} = \frac{4\lambda}{17\lambda - 13\lambda} = 1. \text{ Hence } A = 90°.$$

EXAMPLE 9. *Show that*
$$\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{D^2},$$

SOLUTION. We have
$$\text{L.H.S.} = \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2}$$
$$= \frac{1}{\Delta^2}[s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2]$$
$$= \frac{1}{\Delta^2}[4s^2 - 2(a + b + c)s + a^2 + b^2 + c^2]$$
$$= \frac{a^2 + b^2 + c^2}{\Delta^2} = \text{R.H.S.}$$

EXAMPLE 10. *Show that* $R \geq 2r$ *and that equality holds good iff the triangle is equilateral.*

SOLUTION. The given statement is equivalent to the following:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}\quad\text{with equality iff } A = B = C.$$

We have

$$\sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2}$$
$$=\frac{1}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]\sin\frac{C}{2}$$
$$=\frac{1}{2}\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]\sin\frac{C}{2}$$
$$\le\frac{1}{2}\left[1-\sin\frac{C}{2}\right]\sin\frac{C}{2}$$
$$=\frac{1}{2}\left[\sin\frac{C}{2}-\sin^2\frac{C}{2}\right]$$
$$=\frac{1}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\sin\frac{C}{2}\right)^2\right]\le\frac{1}{8}$$

The equality holds good iff $\cos\frac{A-B}{2} = 1$ and $\sin\frac{C}{2}=\frac{1}{2}$ which happens iff $A = B = C$ (= 60°).

EXERCISE 6.9 C

Prove the following relations for a triangle ABC.

1. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.
2. $rr_1r_2r_3=\Delta^2$.
3. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$.
4. $r_2r_3+r_3r_1+r_1r_2=s^2$.
5. $rr_1+rr_2+rr_3=ab+bc+ca-s^2$.
6. $r_1+r_2+r_3-r=4R$.
7. $r_1^2+r_2^2+r_3^2+r^2=16R^2-a^2-b^2-c^2$.
8. $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$.
9. $a\cot A+b\cot B+c\cot C=2(R+r)$.
10. $(r_2+r_3)(r_3+r_1)(r_1+r_2)=4Rs^2$.
11. $r_1r_2r_3=rs^2$.
12. $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$.
13. $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}+\frac{1}{2R}$.

14. $r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 - 6R(a^2 + b^2 + c^2)$.

15. (The excentral triangle). The triangle formed by joining the excentres I_1, I_2, I_3 of a triangle ABC in pairs is called the *excentral triangle* of triangle ABC. Show that

(*i*) its angles are $\frac{\pi - A}{2}, \frac{\pi - B}{2}, \frac{\pi - C}{2}$;

(*ii*) its sides are $4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}, 4R \cos \frac{C}{2}$;

(*iii*) its circumradius is $2R$;

(*iv*) its inradius is $2R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1\right)$

(*v*) its area is $2Rs$.

16. If the internal bisectors of angles A, B, C, on a triangle meet its circumcircle in A', B', C' respectively, then show that (*i*) A′ is the circumcentre of triangle IBC; (*ii*) the area of triangle $A'B'C'$ is $\frac{1}{2}Rs$.

17. If D, E, F are the points of contact of the incircle of a triangle ABC with its sides then show that

(*i*) the sides of triangle DEF are $2r \cos \frac{A}{2}, 2r \cos \frac{B}{2}, 2r \cos \frac{C}{2}$;

(*ii*) its angles are $\frac{\pi - A}{2}, \frac{\pi - B}{2}, \frac{\pi - C}{2}$;

(*iii*) its area $= Rr \sin A \sin B \sin C = \frac{\Delta r}{2R}$.

18. If d_1, d_2, d_3 are the diameters of the excircles of a triangle ABC, then

$$\frac{a}{d_1} + \frac{b}{d_2} + \frac{c}{d_3} = \frac{d_1 + d_2 + d_3}{a + b + c}.$$

19. If the ex-radii r_1, r_2, r_3 of a triangle are given, explain how to find its sides. Hence find a, b, c given $r_1 = 21, r_2 = 24, r_3 = 28$.

20. If the incircle of a triangle passes through its circumcentre, then show that $\cos A + \cos B + \cos C = \sqrt{2}$.

21. (*a*) Show that r_1, r_2, r_3 and $-r$ are the roots of the equation

$$x^4 - 4Rx^3 + \frac{1}{2}(a^2 + b^2 + c^2)x^2 - \Delta^2 = 0.$$

(*b*) Hence deduce that

(*i*) $r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 - 6R(a^2 + b^2 + c^2)$;

(*ii*) $r_1^4 + r_2^4 + r_3^4 + r^4 = 256R^4 - 32R^2(a^2 + b^2 + c^2)$

$$+ \frac{1}{4}(a^2 + b^2 + c^2) + (b^2c^2 + c^2a^2 + a^2b^2).$$

22. In a triangle ABC, show that

(*i*) $IA.IB.IC = 4Rr^2$.

(*ii*) $II_1.II_2.II_3 = 16R^2r$.

(*iii*) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$.

(*iv*) $I_1A.I_1B.I_1C = 4Rs^2$.

D. The Orthocentre, the Pedal Triangle, the Centroid, the Circumcentre and the Incentre

(*a*) Let ABC be a triangle. We know from Theorem 26 of Chapter 3 that its altitudes are concurrent. The point of concurrence is called the *orthocentre* of the triangle and is denoted by O. The orthocentre falls within or outside the triangle according as the triangle is acute- or obtuse-angled (figures 6.40 and 6.41). If ABC is right-angled, say at A, then its orthocentre O coincides with A, as do the feet of perpendiculars from B and C (figure 6.42). In the first two cases (that is, when triangle ABC is acute — or obtuse-angled) it can also be observed that A, B, C are repectively the orthocentres of triangles

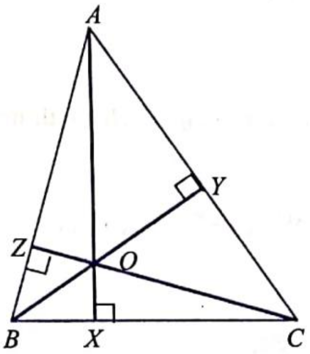


Fig. 6.40

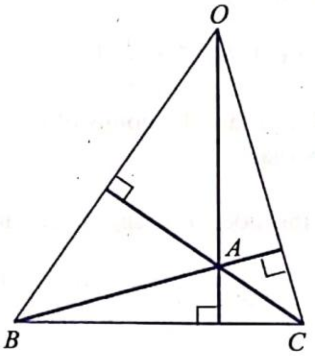


Fig. 6.41

OBC, OCA, OCB. Let us hereafter consider only acute-angled triangles. If the altitudes through A, B, C meet the opposite sides in X, Y, Z respectively, then the triangle XYZ formed by the feet of these altitudes is called *the pedal triangle* of triangle ABC. Also we have six cyclic quadrilaterals namely $OYAZ$. $OZBX$, $OXCY$ $BCYZ$, $CAZX$ and $ABXY$. (see Fig. 6.43) Further the triangles AYZ, BZX, CXY are all similar to triangle ABC. From these facts it can be deduced that

(*i*) the sides of the pedal triangle are $a \cos A = R \sin 2A$; $b \cos B = R \sin 2B$; $c \cos C = R \sin 2C$.

(*ii*) its angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$;

(*iii*) the orthocentre O of triangle ABC is the incentre of its pedal triangle XYZ.

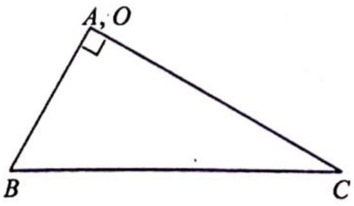


Fig. 6.42

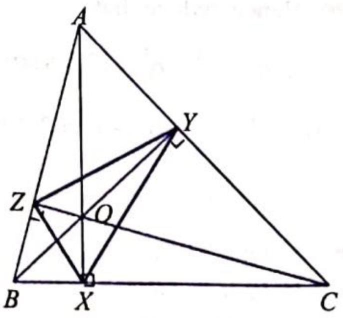


Fig. 6.43

The proofs of these statements are left to the reader. Observe that any triangle ABC is itself the pedal triangle of its excentral triangle and consequently the incentre of a triangle ABC is the orthocentre of its ex-central triangle.

(*b*) If ABC is a triangle and AA_1, BB_1, CC_1, are its medians, we know from Theorem 25 of chapter 3 that these are concurrent at a point usually denoted by G and called the centroid of triangle ABC. It is also known that G divides each median in the ratio 2:1, that is, G is a point of trisection of each of the medians.

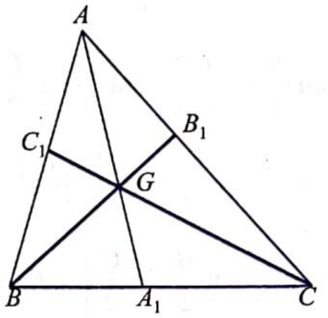


Fig. 6.44

Specifically, $AG : GA_1 = BG : GB_1 = CG : GC_1 = 2:1$. (see Fig. 6.44)

The lengths of the medians are given by $AA_1 = (1/2)\sqrt{2b^2 + 2c^2 - a^2}$, $BB_1 = (1/2)\sqrt{2c^2 + 2a^2 - b^2}$, $CC_1 = (1/2)\sqrt{2a^2 + 2b^2 - c^2}$ (see Problem 2 of Exercise 6.9 B). It is known that from Theorem 56 of Chapter 4 while the centroid G divides the line segment joining the orthocentre O and the circumcentre S in the ratio 2:1, the nine-point centre N divides the same line segment OS in the ratio 1:1, that is, N is the mid-point of OS.

(*c*) Let S be the circumcentre and I the incentre of a triangle ABC. In Fig. 6.45, S is shown as the point of intersection of the perpendicular bisectors of BC and CA, and I as the point of intersection of the internal bisectors of angles A and B. Join SI. We use the cosine rule in triangle ASI to evaluate SI. For this, we need to know AI, AS and $\angle IAS$. If IF is perpendicular to AB, then from triangle IAF,

$$AI = \frac{IF}{(\sin A/2)} = \frac{r}{(\sin A/2)} = 4R \sin(B/2) \sin(C/2).$$

Clearly $AS = R$.

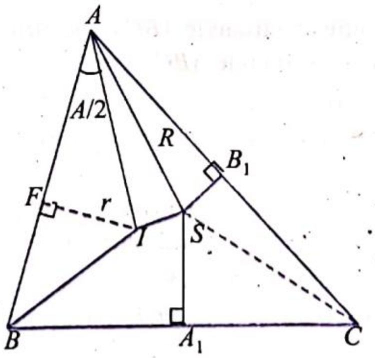


Fig. 6.45

Also $\angle SAB_1 = 90° - \angle ASB_1 = 90° - (1/2)\angle ASC$

$= 90° - \angle ABC = 90° - B.$

So $\angle IAS = \angle IAC - \angle SAC = \dfrac{A}{2} - (90° - B) = \dfrac{B-C}{2}.$

Hence $SI^2 = AS^2 + AI^2 - 2AS \cdot AI \cos \angle IAS$

$$= R^2 + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 2 \cdot R \cdot 4R \sin \frac{B}{2} \sin \frac{A}{2} \cos \frac{B-C}{2}$$

$$= R^2\left[1 + 8 \sin\frac{B}{2} \sin\frac{C}{2}\left(2 \sin\frac{B}{2} \sin\frac{C}{2} - \cos\frac{B-C}{2}\right)\right]$$

$$= R^2\left[1 - 8 \sin\frac{B}{2} \sin\frac{C}{2} . \cos\frac{B+C}{2}\right]$$

$$= R^2\left[1 - 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right] = R^2 - 2Rr.$$

Thus we have proved Euler's theorem (Theorem 4.3 of chapter 4) trigonometrically: $SI^2 = R^2 - 2Rr$.

Similarly, using the cosine rule, we can find expressions for SI_1^2, SO^2, IO^2, I_1O^2 etc.

EXERCISE 6.9 D

1. The altitude through A on BC meets BC in D and the circumcircle of triangle ABC in E when produced. Show that $OD = OE$.
2. Show that the distances from the orthocentre to the vertices of a triangle ABC are $2R \cos A$, $2R \cos B$, $2R \cos C$ and its distances from the sides are $2R \cos B \cos C$, $2R \cos C \cos A$, $2R \cos A \cos B$.
3. Show that in a triangle the distance from the orthocentre to any vertex is twice the distance from the circumcentre to the opposite side.
4. Let ABC be a triangle, O its orthocentre and XYZ be its pedal triangle.

   Show that

   (*a*) the sides of the pedal triangle are $a \cos A$, $b \cos B$, $c \cos C$.

   (*b*) its angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.

   (*c*) its area is $2\Delta \cos A \cos B \cos C$.

   (*d*) its circumradius is $R/2$ and its inradius is $2R \cos A \cos B \cos C$.

   (*e*) its incentre is O.
5. Prove that the circumcentre of a triangle ABC is the orthocentre of the triangle formed by the midpoints of the sides of triangle ABC.
6. Prove that the median through A divides angle A into two parts whose cotangents are $2 \cot A + \cot C$ and $2 \cot A + \cot B$ and makes, with BC an angle whose cotangent is $(1/2)(\cot B - \cot C)$.
7. Four geometrical proofs of the famous Feuerbach's Theorem have been given in Chapter 4 (Theorem 59). Using the methods of this chapter, give a fifth proof.
8. Prove the following for any triangle ABC. (Some of these have already been proved geometrically in Chapter 4. The corresponding references are given in parenthesis)

   (*i*) $SI_1^2 = R^2[1 + 8 \sin(A/2) \cos(B/2) \cos(C/2)] = R^2 + 2Rr_1$.

(*ii*) $IO^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

(*iii*) $SO^2 = R^2(1 - 8 \cos A \cos B \cos C) = 9R^2 - a^2 - b^2 - c^2$.

(Corollary 1 of Theorem 48)

(*iv*) $I_1O^2 = 2r_1^2 - 4R^2 \cos A \cos B \cos C$.

(*v*) $IN = (1/2)R - r$.

(*vi*) $I_1N = (1/2)R + r_1$.

(*vii*) $SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$. (Cor. 3 of Theorem 28)

(*viii*) $AO^2 + BO^2 + CO^2 - SO^2 = 3R^2$ (Cor. 2 of Theorem 48)

(*ix*) $SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = 12R^2$. (Theorems 43, 44)

(*x*) $AN^2 = \frac{1}{4}R^2(1 + 8 \cos A \sin B \sin C) = \frac{1}{4}(R^2 - a^2 + b^2 + c^2)$.

(Cor. of Theorem 57 gives $AN^2 + BN^2 + CN^2$).

(*xi*) $a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = abc$.

(*xii*) $a \cdot AI_1^2 - b \cdot BI_1^2 - c \cdot CI_1^2 = abc$.

(*xiii*) Area of triangle SOI is $2R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$.

9. If any two of the four points S, I, G, O coincide then the triangle is equilateral.

10. If $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius R with $AB = a$, $BC = b$, $CD = c$, $DA = d$, semiperimeter $s = \frac{a+b+c+d}{2}$, then show that

(*i*) $\cos A = \frac{2}{ad+bc}(a^2 + d^2 - b^2 - c^2)$.

(*ii*) $\sin A = \frac{2}{ad+bc}\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

(*iii*) $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

(*iv*) $AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ad+cd}}$,

$BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$. (Brahmagupta's Theorem)

(*v*) $ac + bd = AC.BD$ (Ptolemy's Theorem).

(*vi*) $R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$.

(*vii*) $\tan \frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$.

(*viii*) The product of the segments into which either diagonal is divided by the other is

$$\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}.$$

11. Interpret $\sqrt{\frac{(ab+cd)(cd+bc)}{ac+bd}}$ with reference to the above problem.

12. (*a*) If $ABCD$ is a quadrilateral with the same notation for its sides, semiperimeter and area as in problem 10 above, and $A + C = 2\alpha$, then

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}.$$

(*b*) The area of a quadrilateral is half the product of its diagonals and the sine of the angle between them.

6.10 HEIGHTS AND DISTANCES

In this section, we consider some useful problems in which we calculate the distance between certain points or heights of objects such as towers, buildings, and mountains, which are not directly accessible. Of course we need to know certain other measurements such as lengths and angles, which can be found in practice with the help of instruments.

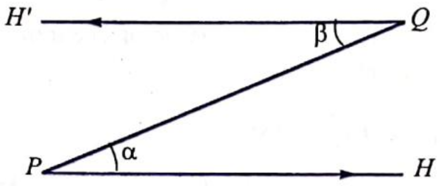


Fig. 6.46

Suppose P and Q are two points in a vertical plane at different horizontal levels. Suppose PH and QH are the horizontal lines in the plane through P and Q as shown in, Fig. 6.46, then angle $HPQ = \alpha$ is called the *angle of elevation* of Q relative to P (or as seen from P) and angle $H'QP = \beta$ is called the *angle of depression* of P relative to Q. Obviously these two angles are equal as the horizontal lines are in the same vertical plane.

EXAMPLE 1. *The angle of elevation of the top of a tower is observed to be 30° from a point on the ground. After walking a distance of 50 metres towards the tower the angle of elevation is found to be 60°. Find the height of the tower.*

SOLUTION. Let PQ be the tower and A and B be first and second points of observation respectively. The whole observation is taking place in one vertical plane as indicated by the data of the problem (A, B, Q are collinear)

Also $\qquad \angle PAQ = 30°, \angle PBQ = 60°, AB = 50$ metres.

Let $\qquad PQ = x$ metres.

From triangles PAQ and PBQ we have

$$AQ = x\cot 30° \quad \text{and} \quad BQ = x\cot 60°$$

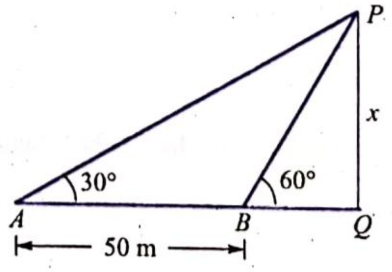


Fig. 6.47

Hence $$x = \frac{AB}{\cot 30° - \cot 60°} = \frac{50}{\sqrt{3} - 1/\sqrt{3}} = 25\sqrt{3}.$$

Thus the height of the tower is $25\sqrt{3}$ metres.

EXAMPLE 2. *From the top of a mountain the angles of depression of three consecutive milestones on a straight road are observed to be α, β, γ respectively. Find the height of the mountain.*

SOLUTION. The point of observation (that is, the peak of the mountain and the mile stones are not in the same vertical plane, in this case. Let P be the point of observation and PQ be the perpendicular drawn from P to the ground. Join P and Q to each of three points A, B, C which represent three collinear mile stones. From the hypothesis $\angle PAQ = \alpha$, $\angle PBQ = \beta$, $\angle PCQ = \gamma$. Let $PQ = h$. Then $AQ = h \cot \alpha$, $BQ = h \cot \beta$, $CQ = h \cot \gamma$. We look at the triangle QAC (in the horizontal plane) for which QB is the median through Q. We apply Appollonius's Theorem to get $QA^2 + QC^2 = 2(QB^2 + AB^2)$. That is, $h^2 \cot^2 \alpha + h^2 \cot^2 \gamma = 2(h^2 \cot^2 \beta + 1)$. Solving for h, we get

$$h = \frac{2}{(\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta)^{1/2}}.$$

Hence the height of the mountain is h miles where h is as given above.

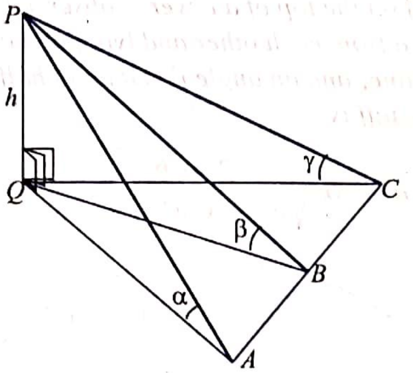


Fig. 6.48

EXAMPLE 3. *A man wishing to ascertain the distance between two objects in a horizontal plane walks along a straight road and observes that at a certain point on the road the two objects subtend the greatest angle α; he walks a distance c along the road, and finds that the objects are in a straight line with his position and that this line makes an angle β with the road. Prove that the distance between the objects is*

$$\frac{2c \sin a \sin b}{\cos a + \cos b}$$

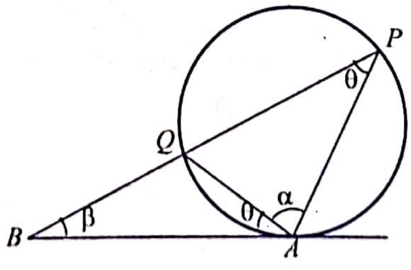


Fig. 6.49

SOLUTION. Let P and Q be the two objects and A and B be the two points of observation. When PQ subtends maximum angle α at A, AB is tangent to the circle passing through P, Q, A. Also $\angle PBA = \beta$ and $AB = c$. If $\angle BAQ = \theta$, then $\angle APB = \theta$ (angle in the alternate segment). Then from ΔPBA,

we get $\qquad \beta + \theta + (\alpha + \theta) = 180°$.

Therefore $\qquad \theta = (180° - \alpha - \beta)/2$. From triangles PQA and BQA, we have

$$\frac{PQ}{\sin\alpha} = \frac{AQ}{\sin\theta} \text{ and } \frac{AB}{\sin(\alpha+\theta)} = \frac{AQ}{\sin\beta}.$$

Dividing the first relation by the second, we get

$$PQ = \frac{c\sin\alpha\sin\beta}{\sin\theta\sin(\alpha+\theta)}$$

$$= \frac{c\sin\alpha\sin\beta}{\sin\left(\frac{\pi}{2}-\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{c\sin\alpha\sin\beta}{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \frac{2c\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}.$$

EXAMPLE 4. *A flagstaff on the top of a tower is observed to subtend the same angle α at two points, distant 2 a from each other and lying in a line through the base of the tower in the horizontal plane; and an angle β at a point half way between them. Prove that the height of the flagstaff is*

$$a\sin\alpha\sqrt{\frac{2\sin b}{\cos a\sin(b-a)}}.$$

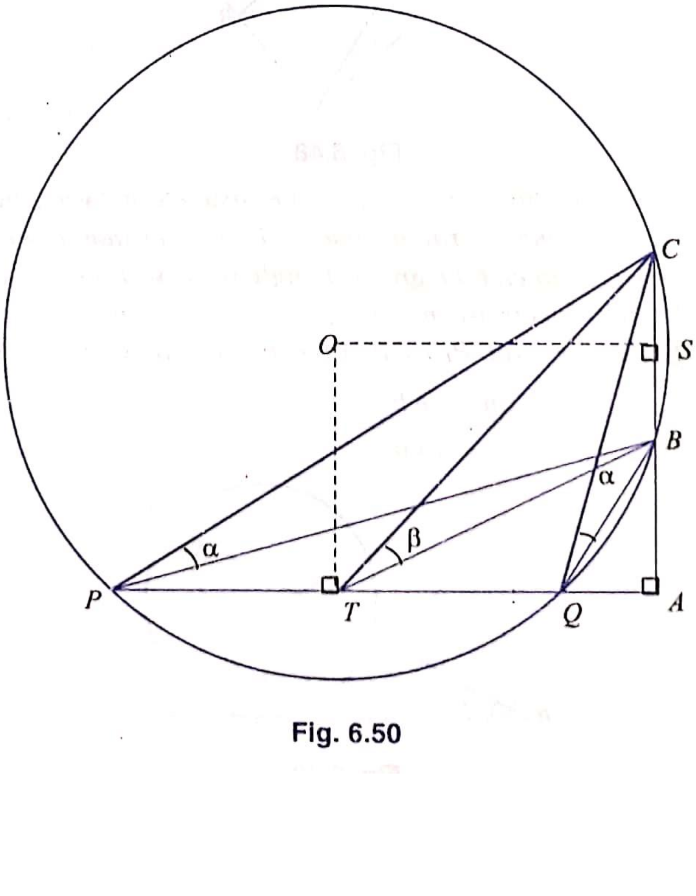


Fig. 6.50

SOLUTION. Let AB be the tower and BC the flagstaff; P, T, Q be the points of observation at which the flagstaff BC subtends angles α, β. a respectively. Since BC subtends equal angles at P and Q, it follows that B, C, P, Q are concyclic. Let O be the centre and r the radius of the circle passing through these four points. Let S be the midpoint of BC. Then $ASOT$ is a rectangle. Let $BC = x$, $AB = y$, $QA = z$. Then from triangle BPC,

$$r = \frac{BC}{2\sin\angle BPC} = \frac{x}{2\sin\alpha}.$$

Now $\angle BOS = (1/2)\angle BOC = (1/2)(2\angle BPC) = \alpha.$

So $TA = OS = BS\cot\alpha = (x/2)\cot\alpha.$

That is, $a + z = (x/2)\cot\alpha.$

If $\angle BTA = \theta$, then $\angle ATC = \beta + \theta$, and so

$$\tan\beta = \tan[(\beta+\theta)-\theta] = \frac{\tan(\beta+\theta)-\tan\theta}{1+\tan(\beta+\theta)\tan\theta}$$

$$\frac{\dfrac{x+y}{a+z}-\dfrac{y}{a+z}}{1+\dfrac{(x+y)y}{(a+z)^2}} = \frac{x(a+z)}{(a+z)^2+(x+y)y}$$

Also from triangles OTQ and OSB,

$$OT^2 + TQ^2 = OQ^2 = OB^2 = OS^2 + SB^2.$$

That is, $(y + x/2)^2 + a^2 = ((x/2)\cot\alpha)^2 + \frac{x^2}{4}.$

Therefore $xy + y^2 = (x^2/4)\cot^2\alpha - a^2.$

Hence
$$\tan\beta = \frac{x(x/2)\cot\alpha}{\left(\frac{x^2}{4}\right)\cot^2\alpha + \left(\frac{x^2}{4}\right)\cot^2\alpha - a^2}$$

Simplifying, we get $x^2 = \dfrac{2a^2\sin\beta\sin^2\alpha}{\sin(\beta-\alpha)\cos\alpha}$, as desired.

EXERCISE 6.10

1. The angle of elevation on the top of a mountain from a point on the ground is found to be α. After walking a distance a along a slope of inclination β towards the cliff, the elevation is found to be γ. Show that the height of the mountain is

$$\frac{a\sin\alpha\sin(\alpha-\beta)}{\sin(\gamma-\alpha)}.$$

2. A man walking on a straight road finds that the line joining two objects on the same side of the road subtends an angle α at some point on the road. After walking a distance b along the road he finds that the line joining the two objects again subtends the same angle α. After walking a further distance a he finds that he is in a line with the objects making an angle α with the road. Find the distance between the two objects.
3. A tower stands vertically in the interior of a field which has the shape of an equilateral triangle of side a. If the angles of elevation of the top of the tower are α, β, γ from the corners of the field find the height of the tower.

4. A pole stands on a horizontal plane inclined to the east at an angle θ to the ground. The elevations of its top from two points due west at distances a and b from the foot of the pole are α and β respectively. Show that

$$\theta = \tan^{-1} \frac{a \cot \beta - b \cot \alpha}{a - b}.$$

5. A ladder is inclined at an angle α to the ground with its top resting on the wall. When the bottom slides through a distance c away from the wall, the inclination of the ladder to the ground is β. Show that the top of the ladder would have descended through a height of

$$c \cos \frac{\alpha - \beta}{2} \div \cos \frac{\alpha + \beta}{2}$$

6. A man observes that when he has walked c metres up an inclined plane, the angular depression of an object in a horizontal plane through the foot of the slope is α, and that when he has walked a further distance of c metres, the depression is β. Prove that the inclination of the slope to the horizontal is $\cot^{-1}(2 \cot \beta + \cot \alpha)$.

7. A flagstaff is on the top of a tower which stands on a horizontal plane. A person observes that the flagstaff and the tower subtend angles α and β at a point on the horizontal plane. He then walks a distance a towards the tower and finds that flagstaff subtends the same angle α. Prove that the height of the tower and the length of the flagstaff are respectively

$$\frac{a \sin \beta \cos(\alpha + \beta)}{\cos(\alpha + 2\beta)} \text{ and } \frac{a \sin \alpha}{\cos(\alpha + 2\beta)}.$$

8. At each end of a horizontal base of length $2a$ it is found that the elevation of a mountain peak is θ and that at the middle point is ϕ. Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{[\sin(\phi + \theta) \sin(\phi - \theta)]^{1/2}}.$$

9. A man walks along a horizontal circle round the foot of a flagstaff, which is inclined to the vertical, the foot of the flagstaff being the centre of the circle. The greatest and least angles which the flagstaff subtends at his eye are α md β; and when he is midway between the corresponding positions the angle is θ. If the man's height is neglected, prove that

$$\tan \theta = \sqrt{\sin^2(\alpha - \beta) + 4 \sin^2 \alpha \sin^2 \beta \sin(\alpha + \beta)}.$$

10. Two lines inclined at an angle γ are drawn on an inclined plane and their inclinations to the horizon are found to be α and β respectively. Show that the inclination of the plane to the horizon is

$$\sin^{-1} \{\operatorname{cosec} \gamma \sqrt{\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \gamma}\}$$

and that the angle between one of the given pair of lines and the line of greatest slope on this inclined plane is

$$\tan^{-1} \left\{ \frac{\sin \beta - \sin \alpha \cos \gamma}{\sin \alpha \cos \gamma} \right\}.$$

6.11 ELIMINATION

Suppose we have two independent simultaneous equations in one unknown quantity. Then generally it is possible to eliminate this unknown between the two equations and get a relation connecting the other parameters in the two equations. For example, suppose we wish to eliminate x from the equations

$$px + q = 0 \qquad (1)$$

and

$$ax^3 + bx + c = 0. \qquad (2)$$

From (1), we have $x = -q/p$. Substituting this in (2), we get

$$a(-q/p)^3 + b(-q/p) + c = 0.$$

That is, $aq^3 + bp^2q - cp^3 = 0$, which is the result of elimination, called the *eliminant.*

Similarly if we have 3 independent equations in 2 unknowns or in general $(n+1)$ independent equations in n unknowns, theoretically we can eliminate the unknowns from the given system and obtain the eliminant. There is no general method which is applicable to all cases (or even majority of the cases) and each problem has to be treated in its own special way. A certain amount of ingenuity is required in some cases to arrive at the eliminant.

EXAMPLE 1. *Eliminate θ from the equations*

$$a \sin^m\theta = b, \quad c \cos^n\theta = d.$$

SOLUTION. From the given equations we have

$$\sin\theta = (b/a)^{1/m} \text{ and } \cos\theta = (d/c)^{1/n}.$$

But $\quad \sin^2\theta + \cos^2\theta = 1$ for all values of θ.

So $\quad (b/a)^{2/m} + (d/c)^{2/n} = 1$, which is the required eliminant.

EXAMPLE 2. *Eliminate θ from the equations*

$$x \sin\theta - y\cos\theta = \sqrt{x^2 + y^2}, \tag{3}$$

$$\frac{\sin^2 q}{a^2} + \frac{\cos^2 q}{b^2} = \frac{1}{x^2 + y^2}. \tag{4}$$

SOLUTION. From (3), squaring both sides, we have

$$x^2 \sin^2\theta - 2xy \sin\theta\cos\theta + y^2\cos^2\theta = x^2 + y^2.$$

Therefore $x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta = 0$,

giving $\quad (x\cos\theta + y\sin\theta)^2 = 0$

Hence $$\frac{\sin\theta}{x} = \frac{\cos\theta}{-y}.$$

So, $$\frac{\sin^2\theta}{x^2} = \frac{\cos^2\theta}{y^2} = \frac{\sin^2\theta + \cos^2\theta}{x^2 + y^2} = \frac{1}{x^2 + y^2}$$

and $$\sin^2\theta = \frac{x^2}{x^2 + y^2}, \quad \cos^2\theta = \frac{y^2}{x^2 + y^2}.$$

Therefore from (4),

$$\frac{x^2}{a^2(x^2 + y^2)} + \frac{y^2}{b^2(x^2 + y^2)} = \frac{1}{x^2 + y^2},$$

yielding $$x^2/a^2 + y^2/b^2 = 1.$$

EXAMPLE 3. *Eliminate α and β from the equations:*

$$\sin\alpha + \sin\beta = l, \tag{5}$$

$$\cos\alpha + \cos\beta = m, \tag{6}$$

$$\tan(\alpha/2)\tan(\beta/2) = n. \tag{7}$$

SOLUTION. From (7), we have

$$\frac{1-n}{1+n} = \frac{1 - \tan(\alpha/2)\tan(\beta/2)}{1 + \tan(\alpha/2)\tan(\beta/2)} = \frac{\cos(\alpha+\beta)/2}{\cos(\alpha-\beta)/2}. \tag{8}$$

Also from (5) and (6),

$$l^2 + m^2 = 2 + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$= 2 + 2\cos(\alpha - \beta) = 4\cos^2(\alpha - \beta)/2.$$

From (6) alone,

$$2m = 4\cos[(\alpha + \beta)/2]\cos[(\alpha - \beta)/2].$$

Hence $$\frac{2m}{(l^2 + m^2)} = \frac{\cos(\alpha + \beta)/2}{\cos(\alpha - \beta)/2}. \quad (9)$$

From (8) and (9), we obtain

$$(l^2 + m^2)(1 - n) = 2m(1 + n).$$

EXERCISE 6.11

Eliminate θ from the equations [(1) – (15)].

1. $a\cos\theta + b\sin\theta = c$; $a\sin\theta - b\cos\theta = d$.
2. $a\cos(\theta + \alpha) = x$; $b\cos(\theta - \beta) = y$.
3. $x\cos\theta - y\sin\theta = \cos 2\theta$; $x\sin\theta + y\cos\theta = 2\sin 2\theta$.
4. $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$; $\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0$.
5. $\lambda\cos 2\theta = \cos(\theta + \alpha)$, $\lambda\sin 2\theta = 2\sin(\theta + \alpha)$.
6. $\cos^2\theta = (m^2 - 1)/3$; $\tan^3(\theta/2) = \tan\alpha$.
7. $2\cos\theta - \cos 2\theta = a$; $2\sin\theta - \sin 2\theta = b$.
8. $x\cos 3\theta + y\sin 3\theta = a\cos\theta$;
   $x\sin 3\theta + y\cos 3\theta = a\cos[\theta + (\pi/6)]$.
9. $\operatorname{cosec}\theta - \sin\theta = a$; $\sin\theta - \cos\theta = b$.
10. $x\cos(\theta + \alpha) + y\sin(\theta + \alpha) = a\sin 2\theta$;
    $y\cos(\theta + \alpha) - x\sin(\theta + \alpha) = 2a\cos 2\theta$.
11. $\sin\theta + \sin 2\theta = x$; $\cos\theta + \cos 2\theta = y$.
12. $\tan\theta - \tan 2\theta = a$; $\cot\theta - \cot 2\theta = b$.
13. $x\sin\theta - y\cos\theta = -\sin 4\theta$;
    $x\cos\theta + y\sin\theta = (5/2) - (3/2)\cos 4\theta$.
14. $\tan(\theta - \alpha) + \tan(\theta - \beta) = x$; $\cot(\theta - \alpha) + \cot(\theta - \beta) = y$.
15. $\frac{\cos(\alpha - 3\theta)}{\cos^3\theta} = \frac{\sin(\alpha - 3\theta)}{\sin^3\theta} = m$.

Eliminate θ and ϕ from the equations [(16) – (21)].

16. $\sin\theta + \sin\phi = x$; $\cos\theta + \cos\phi = y$; $\theta + \phi = \alpha$.
17. $\tan\theta - \tan\phi = a$; $\cot\theta - \cot\phi = b$; $\theta - \phi - \alpha$.
18. $\sin\theta + \sin\phi = a$; $\cos\theta + \cos\phi = b$; $\tan\theta + \tan\phi = c$.
19. $x\cos\theta + y\sin\theta = 1$, $x\cos\theta + y\sin\phi = 1$,
    $p\cos\theta\cos\phi + q\sin\theta\sin\phi = 0$.
20. $\cos\theta + \cos\phi = a$; $\cot\theta + \cot\phi = b$; $\operatorname{cosec}\theta + \operatorname{cosec}\phi = c$.
21. $\cos\theta + \cos\phi = x$; $\cos 2\theta + \cos 2\phi = y$; $\cos 3\theta + \cos 3\phi = 3$.

PROBLEMS

1. If $m^2 + m'^2 + 2mm' \cos\theta = 1$,
$$n^2 + n'^2 + 2nn' \cos\theta = 1,$$
$$mn + m'n' + (m'n + m'n)\cos\theta = 0,$$
then show that $m^2 + n^2 = \operatorname{cosec}^2\theta$.

2. If $\tan(\pi/4 + y/2) = \tan^3(\pi/4 + x/2)$,
then show that $\sin y = \dfrac{3\sin x + \sin^3 x}{1 + 3\sin^2 x}$.

3. If α, β are acute angles and
$[\sin(\alpha - \beta) + \cos(\alpha + 2\beta)\sin\beta]^2 = 4\cos\alpha\cos\beta\sin(\alpha + \beta)$,
then show that $\tan\alpha = \tan\beta\left[\dfrac{1}{(\sqrt{2}\cos\beta - 1)^2} - 1\right]$.

4. Show that
$$\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48° = 1 + \sin^2 9° + \sin^2 18°.$$

5. Prove that $\dfrac{\tan(x + \alpha)}{\tan(x - \alpha)}$ cannot lie between
$$\tan^2(\pi/4 - \alpha) \text{ and } \tan^2(\pi/4 + \alpha).$$

6. Show that $\cos\theta(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$ always lies between $\pm\sqrt{1 + \sin^2\alpha}$.

7. Prove that
$\sin^3(\alpha - \beta)\sin^3(\gamma - \delta) + \sin^3(\beta - \gamma)\sin^3(\alpha - \delta) + \sin^3(\gamma - \alpha)\sin^3(\beta - \delta)$
$= 3\sin(\alpha - \beta)\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \delta)\sin(\beta - \delta)\sin(\gamma - \delta)$.

8. Prove that $\dfrac{\cot 3x}{\cot x}$ never lies between 1/3 and 3.

9. If $\sin x = K\sin(A - x)$, show that
$$\tan(x - A/2) = [(k - 1)/(k + 1)]\tan A/2.$$

10. If in triangle ABC, $\cot A + \cot B + \cot C = \sqrt{3}$, show that the triangle is equilateral.

11. In triangle ABC show that
$$-2 \le \sin 3A + \sin 3B + \sin 3C \le 3\sqrt{3}/2.$$
When does equality hold good on either side?

12. Solve the equation $\cos^n x - \sin^n x = 1$, where n is a positive integer.

13. Solve the equation : $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

14. Find all x in $[0, 2\pi]$, such that
$$2\cos x \le \left|\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}\right| \le \sqrt{2}.$$

15. If in triangle ABC,
$$a + b = \tan(C/2)(a\tan A + b\tan B), \text{ then } a = b.$$

16. If $f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$ and $f(\theta) \ge 0$ for all real θ, then show that $a^2 + b^2 \le 2$, $A^2 + B^2 \le 1$. (Here a, b, A, B are real numbers).

17. If $\tan(A/2)$, $\tan(B/2)$, $\tan(C/2)$ are in A.P. Then so are $\cos A$, $\cos B$, $\cos C$. Here A, B, C are the angles of a triangle.

18. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \theta$ then show that

$$\overset{4}{\underset{i=1}{\pi}} \cos(\theta - \alpha_i) + \overset{4}{\underset{i=1}{\pi}} \sin(\theta - \alpha_i) = \overset{4}{\underset{i=1}{\pi}} \cos \alpha_i + \overset{4}{\underset{i=1}{\pi}} \sin \alpha_i.$$

19. Determine the number of real solutions of $\sin x = x/100$.

20. If $\cot^2 (\beta/2) = \cot(\theta + \alpha)/2 \cdot \cot (\theta - \alpha)/2$, then show that $\cos\theta = \cos \alpha \cos \beta$.

21. If $\sqrt{2} \cos A = \cos B + \cos^3 B$, $\sqrt{2} \sin A = \sin B - \sin^3 B$, prove that $\sin (A - B) = \pm 1/3$.

22. In triangle ABC, M is an interior point on side BC. If r, r', r'' are the inradii and r_1, r'_1, r''_1 are the ex-radii opposite A of triangles ABC, ABM, ACM, then show that

$$\frac{r' r''}{r'_1 r''_1} = \frac{r}{r_1}.$$

23. If x_1, x_2, x_3, x_4 are the roots of the equation
$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - x \sin \beta = 0$,

show that $\sum_{i=1}^{4} \tan^{-1} x_i = n\pi + \pi/2 - \beta$, where n is an integer.

24. If A, B, C are the angles of a triangle, then $\tan^{-1} (\cot B \cot C) + \tan^{-1} (\cot C \cot A) + \tan^{-1} (\cot A \cot B)$

$$= \tan^{-1} \left\{1 + \frac{8 \cos A \cos B \cos C}{\sin^2 2A + \sin^2 2B + \sin^2 2C}\right\}.$$

25. Suppose ABC is an acute triangle. Consider the triangle formed by the three direct (external) common tangents (which are not the sides of triangle ABC) drawn to the excircles of triangle ABC taken pairwise. Find the angles, the sides, the area, the circumradius and the inradius of the triangle so formed.

26. If $ABCD$ is a quadrilateral with $AB + CD = AD + BC$, then show that whether the quadrilateral is convex or not, there is a circle touching all its sides (produced if necessary). When is the radius of the 'incircle' of such a quadrilateral maximum for given lengths of sides?

27. Eliminate α, β, γ from the equations

$$a \cos \alpha + b \cos \beta + c \cos \gamma = 0,$$
$$a \sin \alpha + b \sin \beta + c \sin \gamma = 0,$$
$$a \sec \alpha + b \sec \beta + c \sec \gamma = 0.$$

28. If θ is an angle expressed in radians and $0 < \theta < \pi/2$, then $\sin \theta < \theta < \tan \theta$.

29. Suppose ABC is an acute-angled triangle in a horizontal plane and P, Q, R are three points directly below A, B, C respectively such that $AP = x$, $BQ = x + y$, $CR = x + y + z$. If θ is the angle between the planes containing triangle ABC and triangle PQR, show that $\tan^2 \theta \sin^2 A = y^2/c^2 + z^2/b^2 - 2 (yz/bc) \cos A$.

30. The line joining two objects lying on the same side of a straight road on a horizontal plane subtends two maximal angles α and β at two points on the road distant c from each other. Show that the distance between the objects is
$c \sec[(\alpha + \beta/2] (\sin \alpha \sin \beta)^{1/2}$.

[If the line joining the objects meets the road in P, then on either side of P, there is a point on the road at which the line subtends a maximum angle.]

-

CHAPTER

7

COORDINATE GEOMETRY OF STRAIGHT LINES AND CIRCLES

7.1 INTRODUCTION

Descartes (1596 – 1650) re-created Geometry by using algebraic formulations and methods. The Geometry that arose thus has been called *Cartesian Geometry* for that very reason. It is also called *Analytic Geometry* or *Coordinate Geometry* for reasons which will be obvious in this Chapter. The application of algebra to geometry which became the fashion after Descartes's time may well be named as the key step for all exploration of natural phenomena by mathematics in the past three centuries.

We saw in the first chapter that points on a geometric straight line can be conveniently described by the set of real numbers, once we fix our points corresponding to zero and the number one. We saw through examples that certain geometric problems on straight lines have simple solutions if we describe the points on the line by real numbers and use the algebraic properties of real numbers. Likewise, certain algebraic problems in the set of real numbers have nice geometric solutions via the above correspondence. Suppose now that one wants to describe points on a given plane by means of our familiar real numbers. Imagine that we have a vehicle which can travel in only one direction, but which can move both forward and backward in that direction. If we start from a point, say O on a given plane, we can at best cover all the points on that straight line through O, in the 'direction' of the vehicle. On the other hand, suppose our vehicle can travel in two different directions, say Ox and Oy directions. Of course we assume that the vehicle can go forwards and backwards in the above two direction. If P is any point in the plane, we may draw the lines through P parallel to the given two directions. Let them meet Ox, Oy at A, B respectively. We start from O, move through OA units in the direction of Ox and then AP units in the direction of Oy. This takes us to the point P. In the figure 7.1 that we have drawn, our movements are in the forward direction, both in Ox and Oy directions.

From figures 7.2, 7.3 and 7.4, it is clear that we may have to go forward or backward along the two 'directions' of the vehicle, depending upon the position of P relative to Ox and Oy.

If we have to move x units in the forward Ox direction and y units in the forward Oy direction to reach P from O (as in the case of Fig. 7.1) we may associate the ordered pair (x, y) of real numbers with P. Suppose we have to go x units backwards in the Ox-direction and y units forward in the Oy-direction (as in Fig. 7.2) to reach P, then we associate $(-x, y)$ with P. Similarly $(-x, -y)$ corresponds to P in Fig. 7.3 and $(x, -y)$ to

P in Fig. 7.4. We note that (1, 2) and (2, 1) correspond to two different points in the plane (Fig. 7.5). Although as pairs of real numbers, they are the same they differ in order. Thus, the above discussion enables us to describe points on a plane by the set $\{(x, y) | x, y \text{ are real numbers}\}$ of *ordered pairs of real numbers*, once we fix a point *O* as our origin and two different directions, which we call coordinate directions.

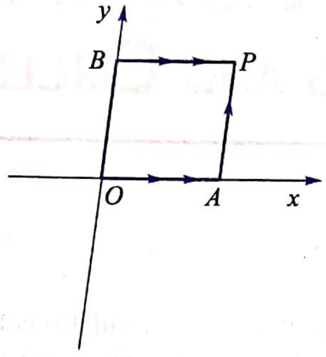


Fig. 7.1

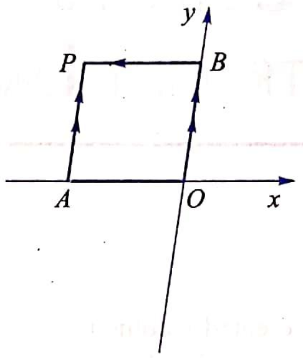


Fig. 7.2

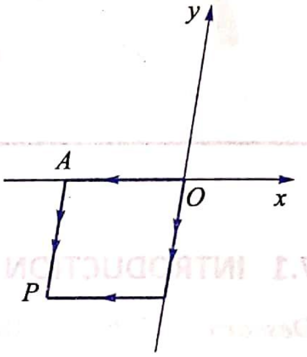


Fig. 7.3

To make things easier, we fix an origin in the plane and two perpendicular directions as our coordinate directions. Let *O* be the origin and $x'Ox$, $y' = Oy$ be the straight lines in the plane along the chosen perpendicular coordinate directions.

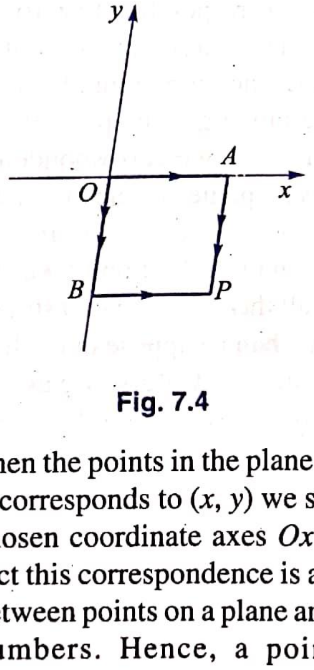


Fig. 7.4

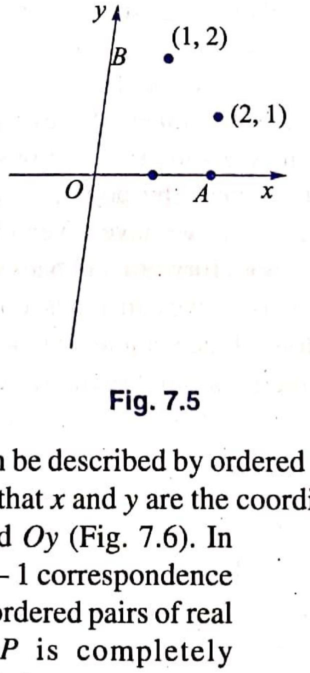


Fig. 7.5

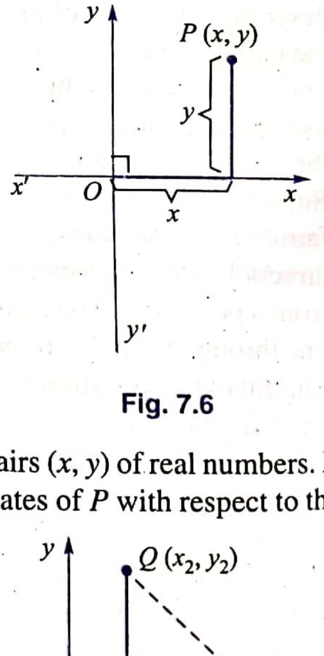


Fig. 7.6

Then the points in the plane can be described by ordered pairs (x, y) of real numbers. If *P* corresponds to (x, y) we say that *x* and *y* are the coordinates of *P* with respect to the chosen coordinate axes *Ox* and *Oy* (Fig. 7.6). In fact this correspondence is a 1 – 1 correspondence between points on a plane and ordered pairs of real numbers. Hence, a point *P* is completely determined by its coordinates (of course, once we fix our origin and coordinate directions).

Fix a pair of rectangular axes $x'Ox$, $y'Oy$ as in Fig. 7.6. Then any point on $x'Ox$ axis is of the form $(x, 0)$ and any point on $y'Oy$ axis is of the form $(0, y)$. We call $x'Ox$ the x – axis and $y'Oy$ the y-axis.

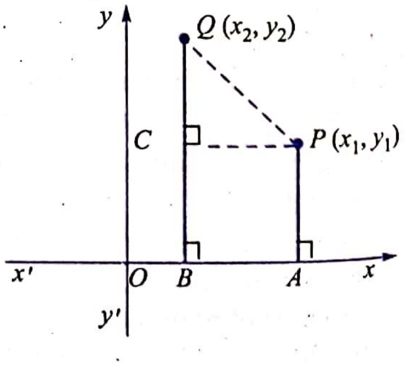


Fig. 7.7

Often in geometry we are interested in distance between points. Consider now two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ with respect to our fixed rectangular axes $x'Ox$ and $y'Oy$. Then $OA = x_1$, $AP = y_1$, $OB = x_2$ and $BQ = y_2$ (see Fig. 7.7). (Note that the coordinates of a point could be positive or negative). From the right angled triangle PCQ we get $PQ^2 = PC^2 + CQ^2 = (OA \pm OB)^2 + (BQ - BC)^2$ (– sign if $x_2 > 0$ and + sign if $x_2 < 0$) $= (x_1 - x_2)^2 + (y_1 - y_2)^2$ (see Figs. 7.7 & 7.8).

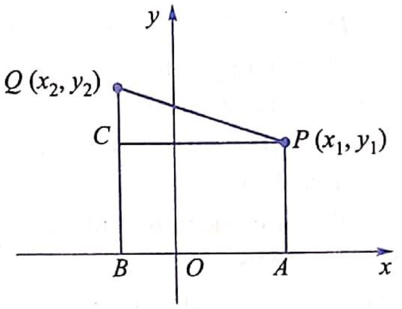


Fig. 7.8

Suppose Q were on the other side of y axis to P as in Fig. 7.8, we still have $BA^2 = (x_1 - x_2)^2$ (note that $|OB| = |x_2|$ and $x_2 < 0$)

In fact, it is not hard to see that, for all positions of $P(x_1, y_1)$ and $Q(x_2, y_2)$ we have $PQ^2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The distance $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ we call the *Euclidean distance* between P and Q. For any point $P(x, y)$ the distance from the origin $O(0, 0)$ is $\sqrt{x^2 + y^2}$.

EXAMPLE 1. The distance between $A(1, 2)$ and $B(-2, 3)$

$$AB = \sqrt{(1-(-2))^2 + (2-3)^2} = 3^2 + (-1)^2 = 10.$$

The distance between $P(\sqrt{2}, \pi)$ and $Q(\pi, \sqrt{3})$

$$PQ = \sqrt{(\sqrt{2} - \pi)^2 + (\pi - \sqrt{3})^2} = \sqrt{2\pi^2 - 2(\sqrt{2} + \sqrt{3})\pi + 5}$$

We note that the distance $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is zero if and only if $(x_1 - x_2)^2 = 0 = (y_1 - y_2)^2$ which happens if and only if $x_1 = x_2$, $y_1 = y_2$. Thus the distance PQ between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is zero if and only if $P(x_1, y_1) = Q(x_2, y_2)$. So, if we want to prove that two points are one and the same, then either we may prove it geometrically or after fixing a pair of coordinate axes, prove that the analytic distance between them is zero. We also note that the distance PQ is the same as the distance QP and that distance PQ + distance $QR \geq$ distance PR for any three points P, Q and R.

EXAMPLE 2. *Show that the triangle whose vertices are $A(-3, -4)$, $B(2, 6)$ and $C(-6, 10)$ is right-angled.*

SOLUTION. We observe that $AB^2 = (-3 - 2)^2 + (-4 - 6)^2 = 125$

$$BC^2 = (2 - (-6)^2 + (6 - 10)^2 = 80$$

$$CA^2 = (-6 - (-3))^2 + (10 - (-4)^2 = 205$$

Thus $CA^2 = 205 = 125 + 80 = AB^2 + BC^2$ and hence $\hat{A}B\hat{C}$ is right angled at B.

EXAMPLE 3. *Prove that the points A(3, – 5), B(– 5, – 4), C(7, 10) and D(15, 9) taken in order are the vertices of a parallelogram.*

SOLUTION. We have

$$AB^2 = (3 - (-5))^2 + (-5 - (-4))^2 = 65$$
$$BC^2 = (-5 - 7)^2 + (-4 - 10)^2 = 340$$
$$CD^2 = (7 - 15)^2 + (10 - 9)^2 = 65$$
$$DA^2 = (15 - 3)^2 + (9 - (-5))^2 = 340.$$

Thus in the quadrilateral $ABCD$, the opposite sides AB, CD and BC, DA are equal in length, which implies that the quadrilateral $ABCD$ is a parallelogram.

EXAMPLE 4. *Find the circumcentre and circumradius of the triangle ABC whose vertices are A(1, 1), B(2, – 1) and C(3, 2).*

SOLUTION. Let $S(x, y)$ be the circumcentre and r be the circumradius of ΔABC. Then $AS^2 = BS^2 = CS^2 = r^2$. This gives

$$(x - 1)^2 + (y - 1)^2 = (x - 2)^2 + (y + 1)^2 = (x - 3)^2 + (y - 1)^2 = r^2.)$$

This in turn gives $-2x - 2y + 2 = -4x + 2y + 5 \Rightarrow 2x - 4y = 3$

$$-4x + 2y + 5 = -6x - 4y + 13 \Rightarrow 2x + 6y = 8$$

Solving, we get $x = 5/2$, $y = 1/2$. Therefore S is $(5/2, 1/2)$ and the circumradius is

$$r = \sqrt{AS^2} = \sqrt{\left(\tfrac{5}{2} - 1\right)^2 + \left(\tfrac{1}{2} - 1\right)^2} = \tfrac{\sqrt{10}}{2} \text{ units.}$$

To find the area of the triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

We have area of ABC = area of trapezium $APRC$

+ area of trapezium $CRQB$

– area trapezium $APQB$ (see Fig. 7.9)

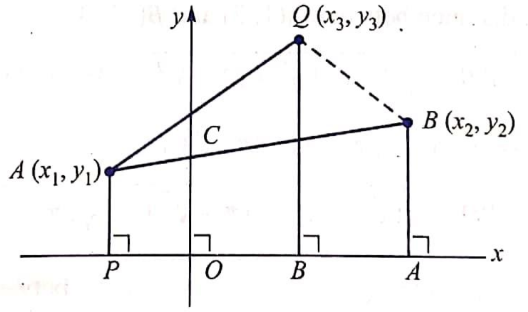


Fig. 7.9

$$= \frac{1}{2}(AP + CR)PR + \frac{1}{2}(CR + QB)RQ - \frac{1}{2}(AP + QB)\,PQ$$

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_2 + y_3)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

$$= \frac{1}{2}\Sigma x_1(y_2 - y_3).$$

We follow the convention the area bounded by an oriented closed curve r is positive if the area enclosed by r lies to the left as one travels on r along its orientation; otherwise the area is negative. Thus the area of $\Delta ABC = -$ area of ΔACB. The area of the triangle

formed by three distinct points A, B and C is zero if and only if they lie on a straight line. Thus a necessary and sufficient condition that three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1)\ x_3(y_1 - y_2)\} = 0$.

EXAMPLE 5. *Find the area of the triangle formed by A(2, 3), B(3, 0) and C(– 4, 2).*

SOLUTION. Applying the formula for the area of a triangle in terms of the coordinates of its vertices, we get

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\{2(0-2) + 3(2-3) + (-4)(3-0)\} = -19/2.$$

Note that we get area of $\Delta ABC < 0$ since the ΔABC has the negative orientation, namely the clockwise orientation and as one travels around ΔABC, the area is to his right.

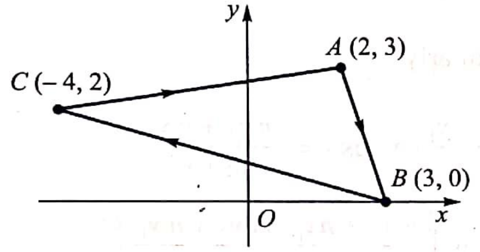


Fig. 7.10

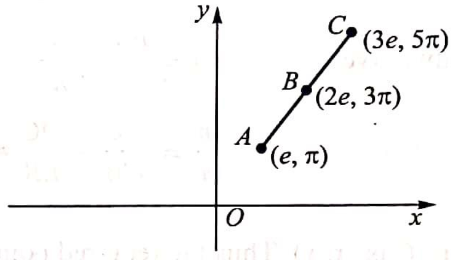


Fig. 7.11

EXAMPLE 6. *Find the area of ΔABC where A is (e, π), B is (2 e, 3 π) and C is (3 e, 5π).*

SOLUTION. Area of ΔABC

$$= \frac{1}{2}\{e\,(3\pi - 5\pi) + 2\,e(5\pi - \pi) + 3\,e(\pi - 3\pi)\}$$

$$= \frac{1}{2}\{-2\pi\, e + 8\pi\, e - 6\pi e\} = 0$$

This implies that A, B, C all lie on a straight line.

Section Formula See Fig. 7.12. We know that given two points A, B there is only one point on the straight line AB which divides the line segment AB in a given ratio. If the given ratio is $\lambda > 0$ then the point of division C lies between A and B such that $AC/CB = \lambda$; on the other hand if $\lambda < 0$, it is external division and the point C lies outside the segment AB such that $AC/CB = \lambda$.

Now let A and B be the points (x_1, y_1) and (x_2, y_2) respectively. Let $\lambda = m/n$ be a given ratio. The problem is to find the coordinates of the point C on the straight line AB such that $AC/CB = m/n = \lambda$.

Case (*i*) $m/n = \lambda > 0$

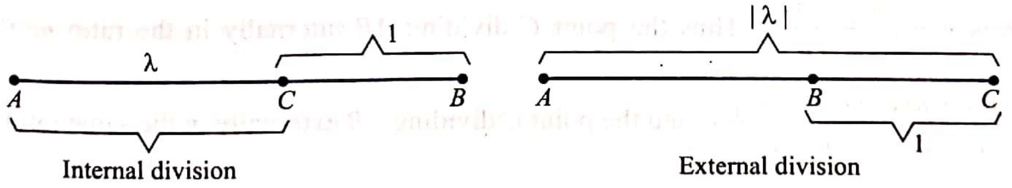


Fig. 7.12

Let C be the point dividing AB in the ratio $m : n$ Then $AC/CB = m/n$. The triangles ADC and CEB are similar (Fig. 7.13). Hence the corresponding sides are proportional.

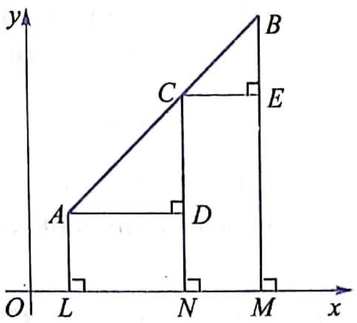


Fig. 7.13

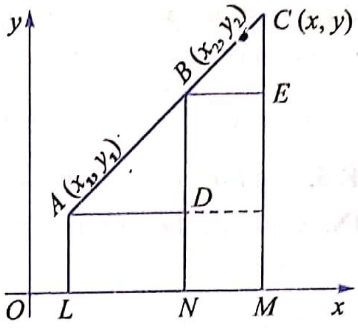


Fig. 7.14

$$\therefore \qquad \frac{m}{n} = \frac{AC}{CB} = \frac{AD}{CE} = \frac{x - x_1}{x_2 - x}$$

and this gives $x = \dfrac{mx_2 + nx_1}{m+n}$. Similarly

$$\frac{m}{n} = \frac{AC}{CB} = \frac{DC}{EB} = \frac{y - y_1}{y_2 - y} \text{ gives } y = \frac{my_2 + ny_1}{m+n}$$

where C is (x, y). Thus the required point C is $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$

Case (*ii*) $\lambda = -m/n = m/-n < 0$ with m, n being positive. Changing n to $-n$ in the above formula we get the corresponding point of division as

$$C = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Also $\dfrac{AC}{CB} = \dfrac{m}{n} < 0$

implies that AC and CB are of opposite orientations and hence C lies outside the segment AB.

The triangles AFC and BEC are similar. Therefore

$$|\lambda| = \left|\frac{-m}{n}\right| = \frac{m}{n} = \frac{AC}{BC} = \frac{AF}{BE} = \frac{x - x_1}{x - x_2}$$

Solving we get $x = \dfrac{mx_2 - nx_1}{m-n}$ and similarly $\dfrac{m}{n} = \dfrac{AC}{BC} = \dfrac{CF}{CE} = \dfrac{y - y_1}{y - y_2}$ where $C(x, y)$ is the required point of division.

gives $y = \dfrac{my_2 - ny_1}{m-n}$. Thus the point C dividing AB internally in the ratio m/n is

$\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$; and the point C dividing AB externally in the same ratio is

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Remark. Any point P on the straight line AB divides the line segment AB in the ratio $AP/PB = \lambda$. The ratio λ is positive when P lies between A and B, and λ is negative when P lies outside the segment AB. (In this case AP and PB are of opposite orientations and hence $AP/PB < 0$). In either case P has coordinates $\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$.

As we vary the parameter λ, we get the various points on the straight line. Thus, the points on the line AB are given by the set $\left\{\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right) : \lambda \text{ is any real number}, \lambda \neq -1\right\}$. When $\lambda = -1$, the point $\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$ is not defined. In other words, there is no point on the straight line AB which divides the line segment AB externally in the ratio 1 : 1.

As an immediate corollary of the section formula, we observe that the midpoint of the line segment AB joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ by taking $\lambda = 1$ in the section formula.

To find the coordinates of the centroid of the triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

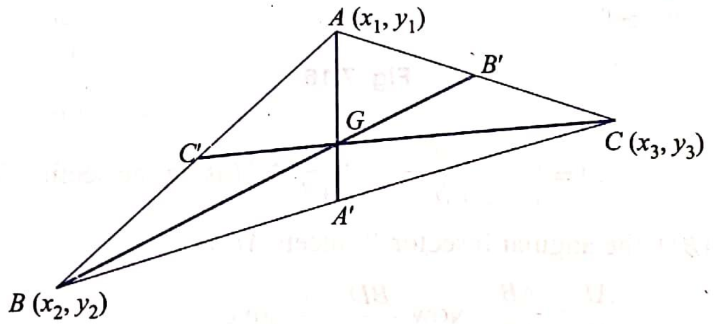


Fig. 7.15

Let A', B'. C' be the midpoints of BC, CA, AB respectively. Then A' is $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$, B' is $\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$ and C' is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

We know that the medians AA', BB' and CC' meet at the centroid G of ΔABC and that G divides each median in the ratio 2 : 1

Therefore $\quad G$ is $\left(\frac{2(x_2 + x_3) + 1.x_1}{2+1}, \frac{2(y_2 + y_3) + 1.y_1}{2+1}\right)$

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Thus the centroid of the triangle with vertices $(x_i, y_i)_{i=1,2,3}$ is the point whose x, y coordinates are the arithmetic averages of x_1, x_2, x_3 and y_1, y_2, y_3 respectively.

Remark. The point dividing AA' in the ratio 2 : 1 was found to be $((x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3$ which is symmetric in x_i and symmetric in y_i. This implies that this point of division also divides the other two medians in the same ratio; this incidentally proves that the medians of a triangle are concurrent and the point of concurrence divides each median in the ratio 2 : 1.

To find the incentre of the triangle *ABC* with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

The incentre I of the triangle ABC is the point of concurrence of the angular bisectors of the internal angles of triangle ABC. Let AD, BE and CF be the internal angular bisectors of ΔABC, meeting the opposite sides at D, E, F (Fig. 7.16). As we have seen in Chapter 3, by the bisector theorem, we have $BD/DC = AB/AC = c/b$; $CE/EA = a/c$ and $AF/FB = b/a$ where a, b, c stand, as usual for the lengths of the sides BC, CA, AB respectively. Hence D has coordinates

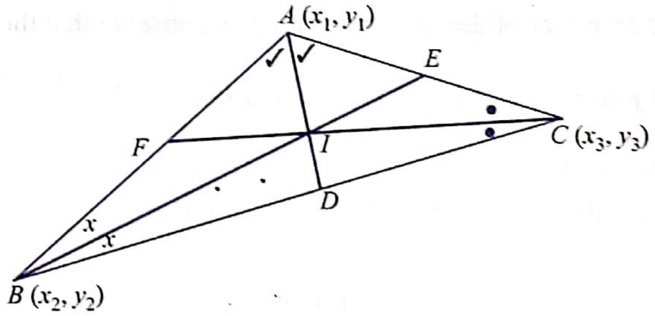


Fig. 7.16

$$D = \left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right) \text{(using the section formula)}$$

Again in ΔABD, the angular bisector BI meets AD at I.

$$\frac{AI}{ID} = \frac{AB}{BD}. \text{ Now } \frac{BD}{DC} = \frac{c}{b} \text{ gives}$$

$$\frac{BD}{BD + DC} = \frac{c}{c+b} \text{ or } \frac{BD}{a} = \frac{c}{c+b}$$

$$\therefore \quad AI/ID = \frac{c+b}{a}.$$

Using the section formula once more we get

$$I = \left(\frac{(c+b)\frac{(cx_3+bx_2)}{c+b} + ax_1}{c+b+a}, \frac{(c+b)\frac{(cy_3+by_2)}{c+b} + ay_1}{c+b+a}\right)$$

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

Remark. We have taken I as the intersection of the bisectors AD and BE and found its coordinates to be $\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$; the symmetry again implies that the other angular bisector CF also passes through I. In other words, this gives another proof of the fact that the internal bisectors of the angles of a triangle are concurrent.

Equation of a Curve In coordinate geometry, we study the geometry of points, straight lines, curves and surfaces using the representation of points by their coordinates, which are real numbers, and using their algebraic and other properties to the maximum extent possible. As we have seen in this introductory section, points in a plane are in one-one correspondence with ordered pairs of real numbers, once we fix our origin and two coordinate directions. The points on a given curve will have their coordinates satisfying certain constraints. In other words, the two coordinates x, y of points on a given curve cannot both be independent. For example, consider the x-axis, namely the straight line $x'Ox$. A point (x, y) lies on this x-axis if and only if $y = 0$. Similarly a point (x, y) will lie on the y-axis if and only if $x = 0$. Again for the straight line L bisecting the angle between the positive coordinate axis, we observe that $P(x, y)$ lies on L if and only if $y = x$. (see Fig. 7.17).

As yet another example, consider the straight line AB where A is $(1, 0)$ and B is $(0, 1)$. The ΔAOB is isosceles and $\angle OAB = \angle OBA = \pi/4$. Let $P(x, y)$ be any point on the straight line AB. If M is the foot of the perpendicular from P on the x-axis, then $OM = x$ and $MP = y$. We have $x + y = OM + MP = OM + MA = OA = 1$ (see Fig. 7.18).

The reader can easily check that $x + y = 1$ wherever be the point P on the staight line AB. Also one can check that if $x + y = 1$ then (x, y) lies on AB. Finally consider the circle C with centre $(0, 0)$ and radius a. If $P(x, y)$ is any point on this circle then its distance from the centre must be the radius a. In other words $\sqrt{(x^2 + y^2)} = a$ or $x^2 + y^2 = a^2$. Conversely, if $x^2 + y^2 = a^2$ then (x, y) is at a distance 'a' from the centre $(0, 0)$ and hence is a point on the circle C.

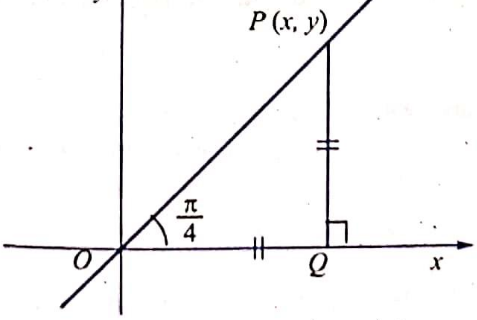


Fig. 7.17

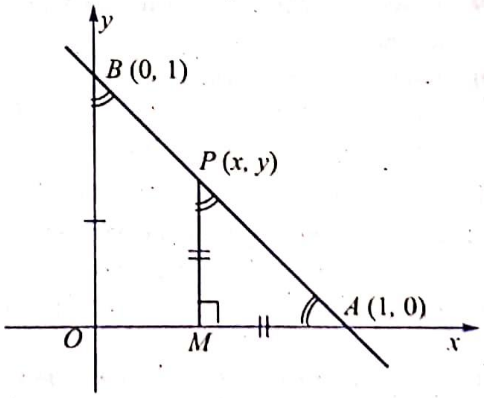


Fig. 7.18

The above examples tell us that points on each of those curves satisfy relations of the form $f(x, y) = 0$ which we call 'the equation of the corresponding curve'. Thus the equation of the x-axis is $y = 0$; the equation of the y-axis is $x = 0$; the equation to the

straight line bisecting the angle between Ox and Oy is $y = x$; the equation of the straight line through (1, 0) and (0, 1) is $x + y = 1$; the equation of the circle with centre (0, 0) and radius a is $x^2 + y^2 = a^2$.

The equation of a curve C is an equation of the form $f(x, y) = 0$ which is satisfied by the coordinates (x, y) of every point on the curve and by no other points.

When a point moves in accordance with certain given conditions, its path is called the *locus of the moving point*. For instance, the locus of a point which moves in a plane such that its distance from a given point A is always a constant r is the circle with centre A and radius r.

EXERCISE 7.1

1. Find the distance between
   (*i*) (2, – 3) and (–3, –6) : (*ii*) (3, 4) and (7, 11) : (*iii*) $(at_1^2, 2at_2)$ and $(at_2^2, 2at_2)$: (*iv*) $(a \cos \theta, b \sin \theta)$ and $(a \cos \phi), b \sin \phi)$: (*v*) $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$.
2. Prove that (4, – 4), (–2, 4) and (6,10) are the vertices of an isosceles triangle.
3. Calculate the lengths of the sides of the triangle whose vertices are (8, 9), (– 4, 4), (4, – 2).
4. Prove that each of the following sets of points forms a rhombus
   (*i*) (2, 5), (6, 2), (2, –1), (–2, 2)
   (*ii*) (3, 4), (–2, 3), (–3, –2), (2, –1).
5. Prove that each of the following sets of points forms a square.
   (*i*) (–3, 1), (–2, –3), (2, –2), (1, 2)
   (*ii*) (0, 2), (3, 8), (9, 5), (6, –1).
6. Calculate the area of the triangle whose vertices are
   (*i*) (2, 4), (7, 9) and (9, 2) (*ii*) (–2, 3), (–7, 5) and (3, –5).
7. Show by area that the following points are collinear:
   (2, 2), (4, – 4), (3, –1).
8. Write down the coordinates of the points dividing the join of (– 8, 3) and (4, 9) in the ratio (*i*) 2:1 (*ii*) 1:5 (*iii*) 4:1 externally.
9. Prove that $P(6, 2)$ is collinear with $A(-2, 2)$ and $B(12, 5)$; find the ratio in which P divides AB.
10. Find the ratio in which the diagonals of the following quadrilaterals $ABCD$ divide one another (*i*) $A(1, 5)$, $B(4, 1)$, $C(7, 5)$, $D(4, 9)$
    (*ii*) $A(10, 10)$, $B(14, 2)$, $C(7, -2)$, $D(2, 2)$.
11. Find the ratios in which the join of $A(-2, 2)$ and $B(4, 5)$ is cut by the axes.
12. Calculate the sides and the perimeter of ΔABC where A is (20, 50), B is (–20, –46) and C is (48, 5). Calculate the area of ΔABC and verify $\Delta^2 = s(s - a)(s - b)(s - c)$.
13. Prove that the four points (– 3, 11), (5, 9), (8, 0) and (6, 8) lie on a circle with center (–1, 2).
14. What are the coordinates of B if $P(3, 5)$ divides the join of $A(-1, 3)$ and B in the ratio 2:3?
15. Prove that $P(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $t : (1 - t)$.
16. Show that (2, –1) is the centre of the circumcircle of ΔABC, where A is (–3, –1), B is (–1, 3) and C is (6, 2). Find the circumradius.

17. Given ΔABC with vertices $A(1, 2)$, $B(8, 4)$ and $C(4, 10)$ find the coordinates of a point P such that ΔPCB, ΔPCA and ΔPAB have the same areas in magnitude and sign.
18. Prove that the lines joining the midpoints of the opposite sides of a quadrilateral bisect one another.
19. Apply Ptolemy's theorem to check that the points (1, –2), (–2, –1), (4, 7) and (6, 3) are concyclic.
20. Apply Ptolemy's theorem as in problem 19 to the four points (–1, –5), (1, –1), (2, 1), (3, 3). Show that the four points are collinear.
21. Show that (7, 5) divides AB and CD in the same ratio where A is (1, 2); B is (5, 4); C is (–5, –1) and D is (3, 3).
22. If G is the centroid of $\Delta\, ABC$, prove that
    (i) $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$
    (ii) $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$ (where O is any point in the plane ABC).
23. Find the incentre of the triangle whose vertices are (0, 0), (20, 15) and (36, 15).
24. In ΔABC, D is the midpoint of BC. Prove that $AB^2 + AC^2 = 2AD^2 + 2DC^2$.
25. If O is the origin and A, B are the points (x_1, y_1) and (x_2, y_2) prove that
    $OA \cdot OB \cos \angle AOB = x_1x_2 + y_1y_2$.
26. A point P moves so that its distance from the point (–1, 0) is always three times its distance from (0, 2). Find the locus of P.
27. If A is $(a, 0)$ and B is $(-a, 0)$ find the locus of P when
    (i) $PA^2 - PB^2 = 2k^2$ = constant.
    (ii) $PA + PB = c$ = constant.
    (iii) $PB^2 + PC^2 = 2PA^2$ where C is $(c, 0)$.

7.2 STRAIGHT LINES

Consider the family of straight lines in the xy-plane. We would like to study this family of straight lines by means of their equations. A straight line is completely determined by any two points on it or by its direction and a point lying on it.

If a straight line L makes an angle θ with the positive x-axis then $\tan\theta$ is defined as the slope of the straight line with respect to the rectangular axes Ox, Oy. By definition, all parallel straight lines have same slope. In fact, two straight lines are parallel if and only if they have the same slope. All straight lines parallel to the x-axis have slope zero; and all straight lines parallel to the y-axis have slope infinity. The straight line $y = x$ bisecting the angle xOy has slope $\tan \pi/4 = 1$.

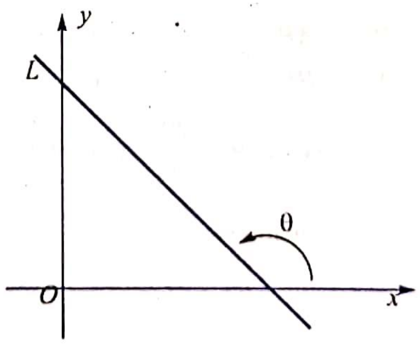


Fig. 7.19

Equation of a straight line passing through two given points.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given two points and let $p(x', y')$ be any point on the straight line. In the adjoining figure, the triangles BRA and ASP are similar.

$$\frac{AS}{BR} = \frac{PS}{AR} \text{ and hence } \frac{x' - x_1}{x_1 - x_2} = \frac{y' - y_1}{y_1 - y_2}$$

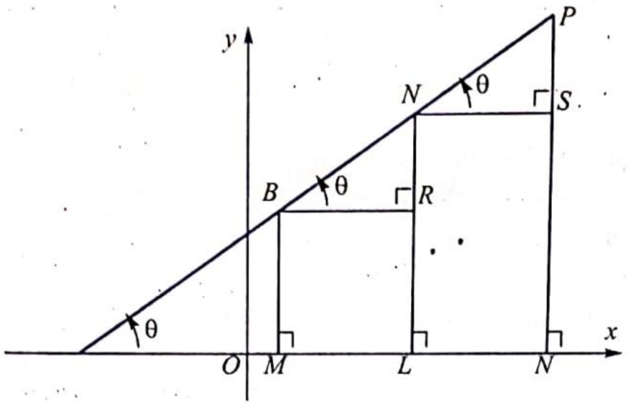


Fig. 7.20

Also conversely if $(x'\ y')$ satisfies $\frac{x' - x_1}{x_1 - x_2} = \frac{y' - y_1}{y_1 - y_2}$ then (x', y') should lie on AB. (Why?) In other words, the equation to the straight line AB joining the given points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is given by $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$.

Note. (1) If P is any point on the straight line AB, then the area of the triangle PAB must be zero. So, if P is (x, y). then

$$0 = x\,(y_1 - y_2) + x_1(y_2 - y) + x_2(y - y_1)$$

This implies that $(x - x_1)\,(y_1 - y_2) = (x_1 - x_2)\,(y - y_1)$

i.e.
$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Also, if area of $\Delta\ PAB = 0$ then P lies on the straight line AB. Therefore the equation to the straight line AB is $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ which is in agreement with what we have got already.

(2) P lies on AB if and only if slope of PA = slope of PB = slope of the straight line $AB = \tan\theta$.

Therefore $\tan\theta = \frac{PS}{AS} = \frac{AR}{BR}$ *i.e.* $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

(see Fig. 7.20 and observe that slope of $PA = \frac{PS}{AS}$ slope of $PB = \frac{AR}{BR}$).

Thus $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ is the equation to the straight line AB. Incidentally, slope of $AB = \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$.

Equation of a straight line in terms of the intercepts it makes with the coordinate axes.

Let L be a straight line making intercepts $OA = a$ and $OB = b$ on the coordinate axes. In our figure (Fig. 7.21). We have $a > 0$, $b > 0$. If $a < 0$, the corresponding point $A\ (a, 0)$ will be on the negative x-axis. Similarly if $b < 0$ the corresponding point $(0, b)$ will be on the negative y-axis. If L makes intercepts a and b on the x and y axes respectively, then L passes through $(a, 0)$ and $(0, b)$. Therefore the equation to the straight line L is

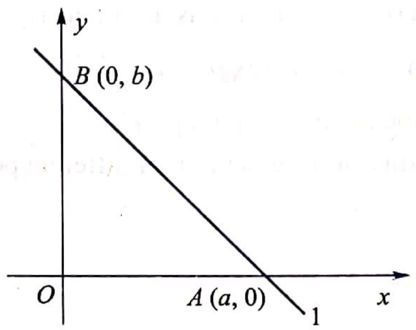


Fig. 7.21

$$\frac{x-a}{a-0} = \frac{y-0}{0-b} \text{ or } \frac{x}{a} - 1 = \frac{y}{b} \text{ or } \frac{x}{a} + \frac{y}{b} = 1.$$

Note. The equation of the straight line L making intercepts 'a' and 'b' can be directly derived, without using the 2 point-formula.

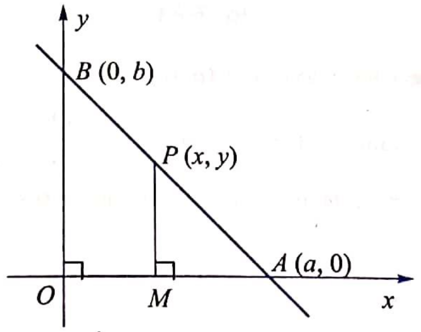


Fig. 7.22

Let $P\ (x, y)$ be any point on AB, and let $PM \perp OX$ as in Fig. 7.22. The triangles AOB and AMP are similar.

$$\frac{OM}{OA} = \frac{BP}{BA} \text{ or } \frac{x}{a} = \frac{PB}{AB}$$

and $$\frac{PM}{BO} = \frac{AP}{AB} \text{ gives } \frac{y}{b} = \frac{AP}{AB}$$

Adding we get $$\frac{x}{a} + \frac{y}{b} = \frac{AP + PB}{AB} = 1.$$

We have taken P between A and B, the reader may check the validity of the equation in all the other cases. Thus, equation of the straight line AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Again, $P(x, y)$ lies on AB if and only if the area of the triangle PAB is zero.

$$0 = x(0 - b) + a(b - y) + 0(y - 0)$$

i.e. $$bx + ay = ab$$

Dividing by ab, $$\frac{x}{a} + \frac{y}{b} = 1.$$

Equation of a straight line in terms of its slope and y-intercept

Let L be the straight line with slope $m = \tan\theta$ and passing through $C(0, c)$, in other words making a y-intercept of 'c' on the y-axis. From the right triangle PNC (Fig. 7.23), we get $\tan(\pi - \theta) = -\tan\theta = -m = CN/NP$. $-m = \dfrac{c - y}{x}$, $y = mx + c$ is the equation to the straight line L with slope m and y-intercept 'c'.

One may check the validity of the equation for different positions of P on the line L.

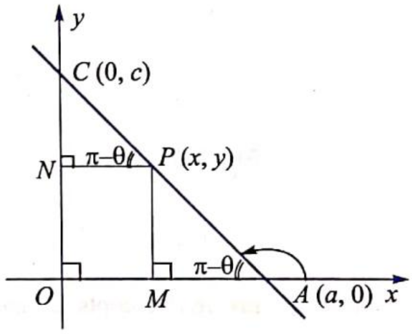


Fig. 7.23

Note. If the above line L meets the x-axis at $A(a, 0)$, then

$$\tan\angle OAC = \tan(\pi - \theta) = -\tan\theta = -m = \frac{OC}{OA} = \frac{c}{a},\ a = -\frac{c}{m}.$$

The line L makes intercepts $a = -c/m$ and c with the x and y axes respectively and therefore its equation is given by

$$\frac{x}{-(c/m)} + \frac{y}{c} = 1 \quad \text{or} \quad y = mx + c.$$

Equation of a straight line in terms of the length of the perpendicular from the origin and the angle which the perpendicular makes with the positive x-axis.

Let the straight line L be at a distance p from $O(0, 0)$, and let the perpendicular from $(0, 0)$ make an angle α with Ox (Fig. 7.24). From the right triangles OAM and OMB we observe that $OA = p \sec\alpha$ and $OB = p$ ccsec α. Using the intercept form of a straight line, the equation to L is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1 \text{ or } x\cos\alpha + y\sin\alpha = p$$

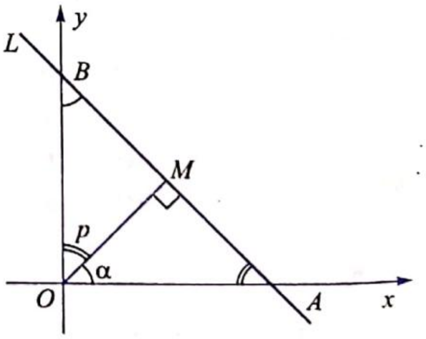


Fig. 7.24

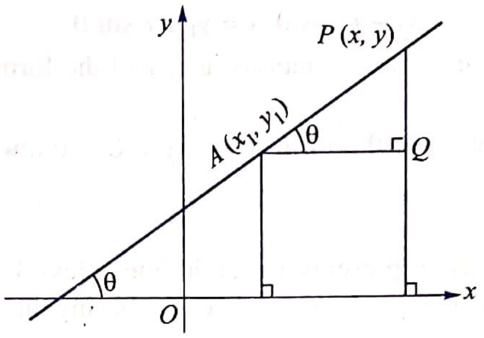


Fig. 7.25

One can easily check that $x \cos \alpha + y \sin \alpha = p$ is satisfied in all possible positions of the straight line.

Equation of a straight line passing through the point (x_1, y_1) and making an angle θ with the positive x-axis.

Let L be the straight line making an angle θ such that $m = \tan \theta$ and passing through $A(x_1, y_1)$. From ΔAQP we get $m = \tan \theta = \dfrac{PQ}{AQ} = \dfrac{y - y_1}{x - x_1}$. This in turn implies that the equation to the straight line L is $y - y_1 = m(x - x_1)$.

Parametric form of a straight line

Let L be the straight line passing through $A(x_1, y_1)$ and making an angle θ with the positive x-axis. From ΔAQP of Fig. 7.25 we get $\cos \theta = \cos \theta = \dfrac{AQ}{AP}$, $\sin \theta = \dfrac{PQ}{AP}$. If we denote the algebraic distance AP as r (for points on L on one side of A the algebraic distance is taken as positive and for points on the other side, the algebraic distance is taken as negative) then

We get $$\frac{AQ}{\cos \theta} = \frac{PQ}{\sin \theta} = r \quad \text{or} \quad \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r.$$

This is the parametric equation of the line L. Any point P on L is of the form $(x_1 + r \cos \theta, y_1 + r \sin \theta)$ where r is the algebraic distance of P from $A(x_1, y_1,)$. $x = x_1 + r \cos \theta$, $y = y_1 + r \sin \theta$ is the parametric equation to L. When one varies r over the real numbers one gets all the points on the straight line L.

We have now derived equation of a straight line in various different forms like.

1. $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2}$ (2 point–formula)
2. $\dfrac{x}{a} + \dfrac{y}{b} = 1$ (intercept form)
3. $y = mx + c$ (Slope–intercept form)
4. $x \cos \alpha + y \sin \alpha = p$ (normal form)
5. $y - y_1 = m(x - x_1)$ (Slope–one point form)
6. $\dfrac{x - x_1}{\cos \theta} = \dfrac{y - y_1}{\sin \theta} = r$ (Parametric form)

$$x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta.$$

All these equations are linear equations in x, y of the form $Ax + By + C = 0$. This prompts the question:

"Does a linear equation of the form $Ax + By + C = 0$ always represent a straight line?"

The answer is yes.

$Ax + By + C = 0$ always represents a straight line unless $A = B = 0$ in which case C also becomes zero. Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points on $Ax + By + C = 0$. Then we have

$$Ax_1 + By_1 + C = 0 \quad (1)$$

$$Ax_2 + By_2 + C = 0 \quad (2)$$

$$Ax_3 + By_3 + C = 0 \quad (3)$$

Multiply the equations (1), (2) and (3) by $y_2 - y_3$, $y_3 - y_1$ and $y_1 - y_2$ respectively and add the resultant equations to get

$$A\,\Sigma x_1(y_2 - y_3) + B\,\Sigma\, y_1(y_2 - y_3) + C\,\Sigma(y_2 - y_3) = 0$$

But $\Sigma\, y_1\,(y_2 - y_3) = 0 = \Sigma(y_2 - y_3)$ and therefore we get

$$A\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} = 0. \quad (4)$$

If $A = 0$ then $Ax + By + C = 0$ becomes $By + C = 0$ which is the straight line $y = -C/B$ parallel to the x-axis, provided $B \neq 0$. If $A = 0 = B$ then $C = 0$. If $A \neq 0$, then the above equation (4) implies that $\Sigma\, x_1\,(y_2 - y_3) = 0$ which in turn implies that the area of the triangle formed by (x_1, y_1), (x_2, y_2) and (x_3, y_3) is zero. This means that any three points on $Ax + By + C = 0$ are collinear; in other words $Ax + By + C = 0$ represents a straight line.

EXAMPLE 1. *Find the equations of the medians of the triangle with vertices at A (1, 2), B (3, 4) and C (– 2, – 5).*

SOLUTION. Let A', B', C' be the midpoints of the sides BC, CA, AB respectively. Then

$$A' \text{ is } \left(\frac{3+(-2)}{2}, \frac{4+(-5)}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

$$B' \text{ is } \left(\frac{-2+1)}{2}, \frac{-5+2}{2}\right) = \left(\frac{1}{2}, -\frac{3}{2}\right)$$

$$C' \text{ is } \left(\frac{1+3}{2}, \frac{2+4}{2}\right) = (2, 3)$$

Equation to the median AA' is $\dfrac{x-1}{1-\frac{1}{2}} = \dfrac{y-2}{2-\left(-\frac{1}{2}\right)}$ which gives on simplification $5x - y - 3 = 0$. Similarly the median BB' has the equation $11x - 7y - 5 = 0$ and the median CC' has the equation $2x - y - 1 = 0$. The centroid G of the given triangle is given by

$$\left(\frac{1+3+(-2)}{3}, \frac{2+4+(-5)}{3}\right) = \left(\frac{2}{3}, \frac{1}{3}\right).$$

One readily checks that the centroid $\left(\frac{2}{3}, \frac{1}{3}\right)$ lies on all the three medians

$$5x - y - 3 = 0$$
$$11x - 7y - 5 = 0$$
$$2x - y - 1 = 0.$$

EXAMPLE 2. *Find the equation of the straight line when the portion of it intercepted between the axes is divided by the point (3, 1) in the ratio 1 : 3.*

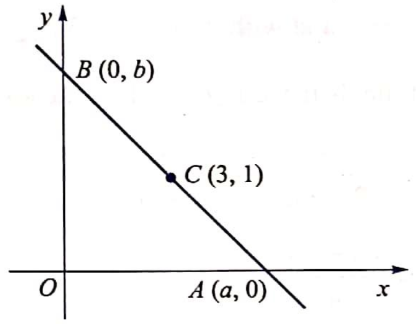


Fig. 7.26

SOLUTION. Let the required straight line meet the x-axis is at A $(a, 0)$ and meet the y-axis at B $(0, b)$. It is given that the point C (3, 1) divides AB or BA in the ratio 1 : 3. Therefore by the section formula, C must be the point

$$\left(\frac{1.0 + 3.a}{4}, \frac{1.b + 3.0}{4}\right) = \left(\frac{3a}{4}, \frac{b}{4}\right) \quad \text{if } \frac{AC}{CB} = \frac{1}{3}$$

or the point $\left(\frac{1.a + 3.0}{4}, \frac{1.0 + 3b}{4}\right) = \left(\frac{a}{4}, \frac{3b}{4}\right) \quad \text{if } \frac{BC}{CA} = \frac{1}{3}$

$$C(3, 1) = \left(\frac{3a}{4}, \frac{b}{4}\right) \text{ gives } a = 4, b = 4 \text{ and } C(3, 1) = \left(\frac{a}{4}, \frac{3b}{4}\right)$$

gives $a = 12$, $b = 4/3$. Hence the required line is either

$$\frac{x}{4} + \frac{y}{4} = 1 \text{ or } \frac{x}{12} + \frac{y}{4/3} = 1 \text{ i.e., either}$$

$$x + y = 4 \text{ or } x + 9y = 12.$$

EXAMPLE 3. *Find the distance of the line $3x - y = 0$ from the point (4, 1) measured along a line making an angle of 135° with the x-axis.*

SOLUTION. The straight line L throgh (4, 1) making an angle of 135° with the x-axis is

$$\frac{x - 4}{\cos 135^\circ} = \frac{y - 1}{\sin 135^\circ} = r. \text{ i.e. } \frac{x - 4}{-1/\sqrt{2}} = \frac{y - 1}{1/\sqrt{2}} = r.$$

(See Fig. 7.27) Any point P on this straight line is of the form $x = 4 - r/\sqrt{2}$, $y = 1 + r/\sqrt{2}$. Where r is the algebraic distance AP. If this point $(4 - r/\sqrt{2}, 1 + r/\sqrt{2})$ were to be on $3x - y = 0$ then $3(4 - r/\sqrt{2}) - (1 + r/\sqrt{2}) = 0$, $11 = 4\, r/\sqrt{2} = 2\sqrt{2}\, r$. Therefore $r = 11/2\sqrt{2} = 11\sqrt{2}/4$ units. Thus the distance of $3x - y = 0$ from (4, 1) measured along L is $11\sqrt{2}/4$ units.

EXAMPLE 4. *Find the equations of the straight lines through the origin whose intercepts between the straight lines $2x + 3y = 12$ and $2x + 3y = 15$ are each equal to three.*

SOLUTION. Assume the equation of the required straight line to be $y = mx$. Here m is to be determined. The intersection of this with $2x + 3y = 12$ is

$$p = \left(\frac{12}{2+3m}, \frac{12m}{2+3m}\right) \text{ and with } 2x + 3y = 15 \text{ is } Q = \left(\frac{15}{2+3m}, \frac{15m}{2+3m}\right).$$

The requirement is that the distance $PQ = 3$. This means

$$9 = \left(\frac{3}{2+3m}\right)^2 + \left(\frac{3m}{2+3m}\right)^2$$

which leads to $$m = \frac{-3 \pm \sqrt{3}}{4}.$$

Thus the required lines are

$$y = \frac{-3+\sqrt{3}}{4}x \quad \text{and} \quad y = \frac{-3-\sqrt{3}}{4}x.$$

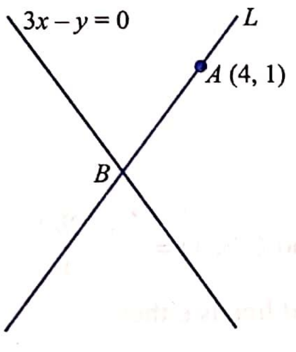


Fig. 7.27

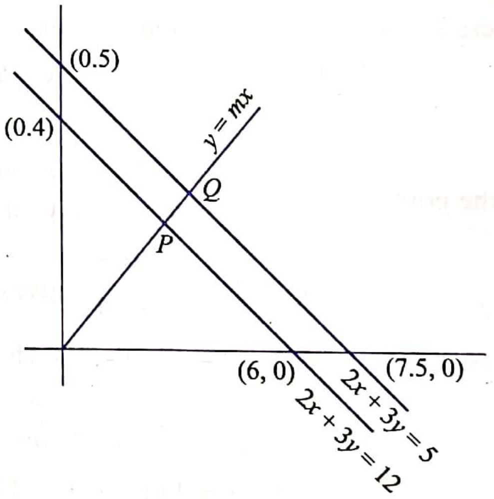


Fig. 7.28

Angle between straight lines

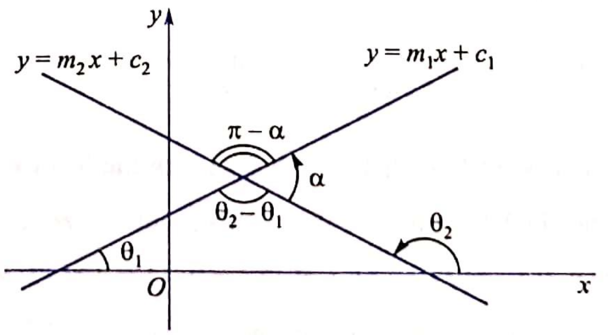


Fig. 7.29

Consider two straight lines $y = m_1x + C_1$ and $y = m_2x + C_2$. Then the two angles between these straight lines are supplementary angles. From Fig. 7.29 it is clear that the two angles between the two straight lines are $\theta_2 - \theta_1$ and $\pi - (\theta_2 - \theta_1)$ where θ_1 and θ_2 are the angles made by the given lines with the positive x-axis. Therefore we have $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$ and the angles between the straight lines are given by the equation

$$\tan\theta = \pm\tan(\theta_1 - \theta_2) = \pm\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

If $\frac{m_1 - m_2}{1 + m_1m_2} > 0$ then $\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$ gives the acute angle between the two straight lines and if $\frac{m_1 - m_2}{1 + m_1m_2} < 0$ then $\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$ gives the obtuse angle between the straight lines.

Note. (1) Two straight lines are parallel iff their slopes are equal. The straight line $Ax + By + C = 0$ has slope $-A/B$ (compare it with $y = mx + C$) and hence two straight lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel iff $b_1/A_1 = b_2/A_2$ or equivalently $A_1/A_2 = B_1/B_2$; in which case $A_2 = kA_1$, and $B_2 = kA_2$ for some constant k. This makes $A_2x + B_2y + C_2 = 0$ as $k(A_1x + B_1y + C_2/k) = 0$ which is the same as $A_1x + B_1y + C_2/k = 0$. Thus any two parallel straight lines can be put in the form $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ so that they differ only in the constant terms.

(2) Two straight lines $y = m_1x + C_1$ and $y = m_2x + C_2$ are perpendicular if and only if $\frac{m_1 - m_2}{1 + m_1m_2} = \tan\pi/2 = \infty$ which happens if and only if $m_1m_2 = -1$. Thus two straight lines are perpendicular if and only if the product of their slopes is -1. The straight line $ax + by + c = 0$ has slope $-a/b$ and hence any straight line perpendicular to it must have slope b/a. Therefore any straight line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + C' = 0$ for some constant C'.

EXAMPLE 5. *An equilateral triangle has its centroid at the origin and one side is $x + y = 1$. Find the other sides of the triangle.*

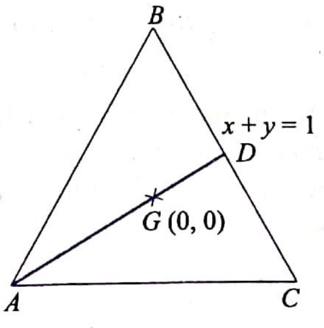


Fig. 7.30

SOLUTION. Let the vertex not on the line $x + y = 1$ be $A(x_1, y_1)$. Let the other two vertices be $B(b, 1 - b)$ and $C(c, 1 - c)$. Note that we have here used the fact that B and C lie on $x + y = 1$. We are given that G is (0, 0). Let AG meet $x + y = 1$ at D. Since G is the centroid, we have

$$b + c + x_1 = 0 = 1 - b + 1 - c + y_1 \qquad (1)$$

But $AD \perp BC$ ($\because$ the triangle is equilateral). $\therefore$ 'm' of $AD \times$ 'm' of $BC = -1$

i.e., $$\frac{y_1}{x_1} \times (-1) = -1 \text{ so that } y_1 = x_1.$$

Substituting this in (1) we get $b + c = 1$. This gives $c = 1 - b$. Thus the three vertices of the triangle are $A(x_1, x_1)$, $B(b, 1 - b)$ and $C(1 - b, b)$. Again equation (1) gives $x_1 + b + 1 - b = 0$, which means $x_1 = -1$. Hence A is $(-1, -1)$.

Now equation to AC is

$$y + 1 = \frac{b+1}{2-b}(x+1)$$

which reduces to

$$y = \frac{b+1}{2-b}x + \frac{2b-1}{2-b} \tag{2}$$

Similarly, equation to AB is

$$y = \frac{2-b}{b+1}x + \frac{1-2b}{b+1} \tag{3}$$

Now we have only to find b. We know AC and BC make an angle of 60°. So

$$\sqrt{3} = \frac{\frac{b+1}{2-b}+1}{1+\left(\frac{b+1}{2-b}\right)(-1)} \tag{*}$$

This gives, $b = -\dfrac{\sqrt{3}-1}{2}$. Since this gives a positive value for the 'm' of AC (check !), we accept this value and do not proceed with $-\sqrt{3}$ on the L.H.S. of (*). Substituting the value of b in (2) and (3) we get the required lines.

EXAMPLE 6. *If the image of the point (h_1, k_1) with respect to the line $ax + by + c = 0$ is the point (h_2, k_2) then show that*

$$\frac{h_2 - h_1}{a} = \frac{k_2 - k_1}{b} = -2\,\frac{(ah_1 + bk_1 + c)}{a^2 + b^2}$$

SOLUTION. As $Q\,(h_2, k_2)$ is the image of $P(x_1, y_1)$ with respect to the line $ax + by + c = 0$ we must have PQ perpendicular to $ax + by + c = 0$ and $PR = RQ$ (see Fig. 7.31) where R is the point of intersection of PQ with the straight line $ax + by + c = 0$.

$\therefore$ Slope $PQ \times$ Slope $(ax + by + c) = -1$

i.e., $$\frac{k_2 - k_1}{h_2 - h_1} \times \frac{-a}{b} = -1$$

This implies that $\dfrac{k_2 - k_1}{b} = \dfrac{h_2 - h_1}{a} = \lambda$ (say).

Then $h_2 = h_1 + a\lambda$ and $k_2 = k_1 + b\lambda$. This gives the midpoint R of PQ as

$$\left(\frac{h_1 + h_1 + a\lambda}{2}, \frac{k_1 + k_1 + b\lambda}{2}\right).$$

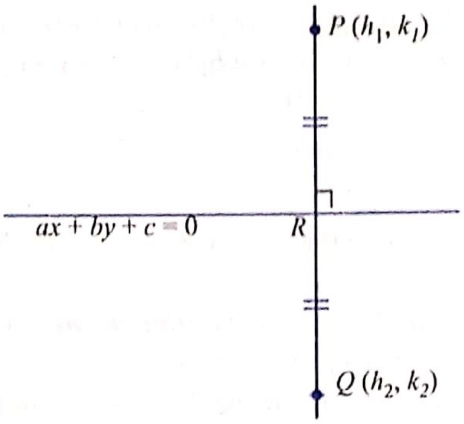


Fig. 7.31

$\therefore$ R is the point $\left(\dfrac{2h_1 + a\lambda}{2}, \dfrac{2k_1 + b\lambda}{2}\right)$.

Now R lies on $ax + by + c = 0$ gives

$$a\left(\frac{2h_1 + a\lambda}{2}\right) + b\left(\frac{2k_1 + b\lambda}{2}\right) + c = 0$$

$\therefore$ $$(a^2 + b^2)\lambda = -2(ah_1 + bk_1 + c)$$

or $$\lambda = -2\,\frac{(ah_1 + bk_1 + c)}{a^2 + b^2}$$

Thus $$\frac{h_2 - h_1}{a} = \frac{k_2 - k_1}{b} = \lambda = -2\,\frac{ah_1 + bk_1 + c}{a^2 + b^2}.$$

Equation of a family of straight lines passing through the intersection of two given lines.

Consider two straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv a_2x + b_2y + c_2 = 0$ intersecting at a point $P\ (h, k)$. Now

$$L_1 + \lambda L_2 \equiv (a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0 \qquad (*)$$

is again a linear equation, namely

$$(a_1 + \lambda a_2)\,x + (b_1 + \lambda\,b_2)\,y + (c_1 + \lambda\,c_2) = 0$$

and hence is also a straight line. Further the point (h, k) satisfies $(a_1h + b_1k + c_1) + \lambda$ $(a_2h + b_2k + c_2) = 0 + \lambda.\,0 = 0$. Therefore, (*) is a straight line passing through the point of intersection of L_1 and L_2. Conversely, suppose L is any straight line passing through the point of intersection of L_1 and L_2. Let $px + qy + r = 0$ be the straight line L.

Then L can be written as $y - k = m(x - h)$ where m is its slope. Solving $a_1h + b_1k + c_1 = 0$, $a_2h + b_2k + c_2 = 0$ we get

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

$\therefore$ $y - \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = m\left(x - \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right)$ is the equation of L.

Simplifying we get $m\,(a_1b_2 - a_2b_1)\,x - y\,(a_1b_2 - a_2b_1) + (a_2c_1 - a_1c_2) - m\,(b_1c_2 - b_2c_1) = 0$ *i.e.*, $(a_1 + mb_2)\,(a_1x + b_1y + c_1) - (a_1 + mb_1)\,(a_2x + b_2y + c_2) = 0$ which is of the form $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$

where $$\lambda = -\frac{a_1 + mb_1}{a_2 + mb_2}$$

Note. When we very λ in $L_1 + \lambda L_2 = 0$ we get the family of straight lines passing through the intersection of $L_1, = 0$ and $L_2 = 0$.

EXAMPLE 7. *The equation of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the orthocentre of the triangle.*

SOLUTION. Let AB be $x + 2y = 0$, BC be $4x + 3y = 5$ and CA be $3x + y = 0$. The slope of BC is $-4/3$ and therefore the slope of AD is $3/4$. Hence the equation to AD is $3x - 4y = 0$. Similarly one finds that the altitude BE must be of the form $x + 2y + \lambda\,(4x + 3y - 5) = 0$ or $(1 + 4\lambda)\,x + (2 + 3\lambda)\,y - 5\lambda = 0$.

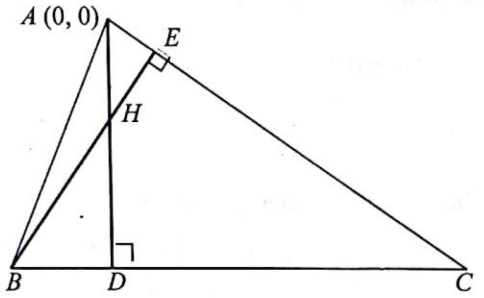


Fig. 7.32

$$-1 = (\text{Slope of } BE)\,(\text{Slope of } CA) = -\left(\frac{1+4\lambda}{2+3\lambda}\right)(-3) \text{ or } 3(1 + 4\lambda) = -(2 + 3\lambda).$$

This gives $\lambda = -1/3$ or BE is given by $x - 3y - 5 = 0$.

The orthocentre H of $\Delta\,ABC$ is got by solving $3x - 4y = 0$, $x - 3y = 5$. We get H as $(-4, -3)$.

EXAMPLE 8. *Prove that the diagonals of the parallelogram formed by the lines $ax + by + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c = 0$, $a'x + b'y + c' = 0$ will be at right angles if $a^2 + b^2 = a'^2 + b'^2$.*

SOLUTION. Now the diagonal AC is of the form $ax + by + c + \lambda\,(a'x + b'y + c) = 0$ as it passes through the intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ (Fig. 7.33).

$$\text{Slope of } AC = -\left(\frac{a + \lambda a'}{b + \lambda b'}\right)$$

Similarly, the diagonal Ac is also of the form $(ax + by + c') + \mu\,(a'x + b'y + c' = 0)$. Thus AC is given by $(a + \lambda\,a')\,x + (b + \lambda\,b')\,y + (1 + \lambda)\,c = 0$ and $(a + \mu\,a')\,x + (b + \mu\,b')\,y + (1 + \mu)\,c' = 0$.

We must have $$\frac{a + \lambda a'}{a + \mu a'} = \frac{b + \lambda b'}{b + \mu b'} = \left(\frac{1+\lambda}{1+\mu}\right)\frac{c}{c'}$$

$$\lambda(a'b - ab') = \mu(a'b - ab')$$

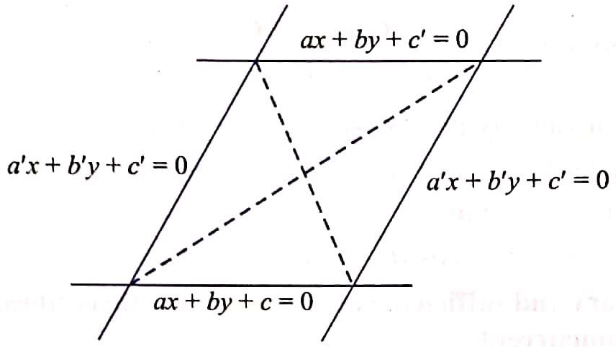


Fig. 7.33

Now $\frac{a}{b} \neq \frac{a'}{b'}$ (Why?) and hence $\lambda = \mu$. This means that $1 = \frac{1+\lambda}{1+\lambda\frac{c}{c'}}$.

But $c \neq c'$ (why?) and hence $\lambda = -1$. Thus $\lambda, = k = -1$.

Therefore equation to AC is $(a - a')x + (b - b')y = 0$. Similarly equation to BD is $(a + a')x + (b + b')y + c + c' = 0$. (It is easily observed that $(ax + by + c) - (a'x + b'y + c) = 0$ passes through A and C; and $(ax + by + c) + (a'x + b'y + c') = 0$ passes through B and D).

Now AC is perpendicular to BD if and only if the product of their slopes is -1; which happens iff

$$\left(-\frac{a-a'}{b-b'}\right)\cdot\left(-\frac{a+a'}{b+b'}\right) = -1 \text{ or simplifying } a^2 + b^2 = a'^1 + b'^2$$

Remark. If the sides of parallelogram are $L_1 \equiv ax + by + c = 0$, $L_2 \equiv a'x + b'y + c = 0$, $L_3 \equiv ax + by + c' = 0$ and $L_4 \equiv a'x + b'y + c' = 0$, the diagonals are given by $L_1 - L_2 = 0$ and $L_1 + L_2 = 0$.

EXAMPLE 9. *The sides of a triangle are $U_r \equiv x \cos \alpha\, r + y \sin \alpha\, r - p_r = 0$ for $r = 1, 2, 3$.*

Show that its orthocentre is given by

$$U_1 \cos(\alpha_2 - \alpha_3) = U_2 \cos(\alpha_3 - \alpha_1) = U_3 \cos(\alpha_1 - \alpha_2).$$

SOLUTION. Let BC be $U_1 \equiv x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0_1$,

CA be $U_2 = 0$ and AB be $U_3 = 0$.

Then the altitude AD through A is of the form $U_2 + \lambda U_3 = (x \cos \alpha_2 + y \sin \alpha_2 - p_2) + \lambda(x \cos \alpha_3 + y \sin \alpha_3) - p_3 = 0$ *i.e.*, AD is given by $(\cos \alpha_2 + \lambda \cos \alpha_3)x + (\sin \alpha_2 + \lambda \sin \alpha_3)y - (p_2 + \lambda p_3) = 0$.

$$\text{Slope of } AD = -\frac{\cos\alpha_2 + \lambda\cos\alpha_3}{\sin\alpha_2 + \lambda\sin\alpha_3}. \quad \text{Now } AD \perp BC \text{ gives}$$

$$\left(-\frac{\cos\alpha_2 + \lambda\cos\alpha_3}{\sin\alpha_2 + \lambda\sin\alpha_3}\right)(-\cot\alpha_1) = -1,$$

or $\quad \cos\alpha_1(\cos\alpha_2 + \lambda\cos\alpha_3) + \sin\alpha_1(\sin\alpha_2 + \lambda\sin\alpha_3) = 0,$

This gives $\quad (\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2 + \lambda(\cos\alpha_1\cos\alpha_3 + \sin\alpha_1\sin\alpha_3) = 0$

or $\quad$ equivalents $-\lambda = -\frac{\cos(\alpha_1 - \alpha_2)}{\cos(\alpha_3 - \alpha_1)}.$

$\therefore$ Equation to AD is $U_2 - \dfrac{\cos(\alpha_1 - \alpha_2)}{\cos(\alpha_3 - \alpha_1)U_3 = 0}$

or equivalently $U_2 \cos(\alpha_3 - a_1) = U \cos(\alpha_1 - \alpha_2)$.

Similarly the altidude BE is $U_3 \cos(\alpha_1 - \alpha_3) = U_1 \cos(\alpha_2 - \alpha_3)$.

$\therefore$ The orthocentre is given by

$$U_1 \cos(\alpha_2 - \alpha_3) = U_2 \cos(\alpha_3 - \alpha_1) = U_3 \cos(\alpha_1 - \alpha_2).$$

To find the necessary and sufficient condition that the three lines $a_i x + b_i y + c_i = 0$, $i = 1,2,3$ may be concurrent.

Solving $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

We get $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

Now this point lies on the third line $a_3x + b_3y + c_3 = 0$ iff

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

or $\quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

See chapter 8 for a standard way of arriving at this equation.

Another necessary and sufficient condition for the above three lines to be concurrent is the following; viz., There exist three constants k_1, k_2, k_3 not all zero such that $k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) + k_3(a_3x + b_3y + c_3)$ is identially zero.

Proof. Suppose there exist k_1, k_2, k_3 not all zero such that $k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) + k_3(a_3x + b_3y + c_2) \equiv 0$. We may assume that $k_3 \neq 0$. Then the above condition gives

$$k_1/k_3(a_1x + b_1y + c_1) + k_2/k_3(a_2x + b_2y + c_2) + a_3x\ b_3y + c_3 \equiv 0 \qquad (1)$$

If (h, k) is the point of intersection of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, substituting in (1) we see that $a_3h + b_3k + c_3 = 0$. Hence (h, k) lies on the third line $a_3x + b_3y + c_3 = 0$. In other words the three line are concurrent.

Conversely, if the three lines are concurrent, we can write $a_3x + b_3y + c_3 = 0$ in the form $(a_1x - b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$ for some constant λ. This means that as $a_3 = k(a_1 + \lambda a_2)$,

$b_3 = k(b_1 + \lambda b_2)$ $c_3 = k(c_1 + \lambda c_2)$ for some constant k. Therefore, we get

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) - \frac{1}{k}(a_3x + b_3y + c_3) \equiv 0$$

The proof is now complete. ❑

EXAMPLE 10. *Show that the straight lines $2x + 7y + 27 = 0$, $5x + 13y - 17 = 0$ and $12x + 33y - 7 = 0$ are concurrent.*

SOLUTION. We note that $(2x + 7y + 27) + 2(5x + 13y - 17) - (12x + 33y - 7) \equiv 0$. Hence the three straight lines are concurrent.

To find the ratio in which the straight line $ax + by + c = 0$ divides the line joining (x_1, y_1) and (x_2, y_2).

Suppose the straight line joining $A(x_1\ y_1)$ and $B(x_1\ y_2)$ meets $ax + by + c = 0$ at $P(x, y)$.

Let $\frac{AP}{PB} = \lambda$

Then P has coordinates,

$$\frac{\lambda x_2 + x_1}{\lambda + 1}, \quad \frac{\lambda y_2 + y_1}{\lambda + 1}$$

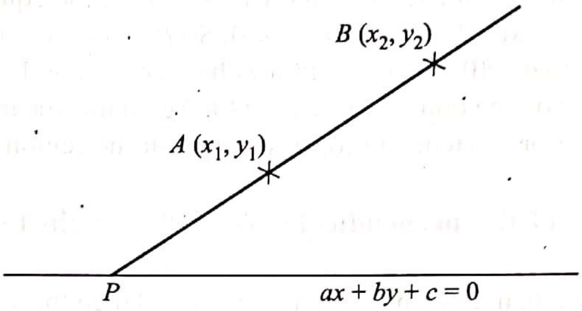


Fig. 7.34

P lies on $ax + by + c = 0$ gives $a(\lambda x_2 + x_1) + b(\lambda y_2 + y_1) + c(\lambda + 1) = 0$

or $\qquad \lambda(ax_2 + by_2 + c) = -(ax_1 + by_1 + c)$.

$\therefore \qquad \lambda = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. This is the required ratio.

Remark. The above ratio is positive if and only if A and B are on the opposite sides of $ax + by + c = 0$. Therefore $A(x_1\ y_1)$ and $B(x_2\ y_2)$ are on the same side of $ax + by + c = 0$ if and only if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign.

EXAMPLE 11. *Show that the origin is within the triangle whose vertices are A(2, 1),. B(3, – 2) and C(– 4, – 1).*

SOLUTION. We have the sides BC, CA, AB having the equations

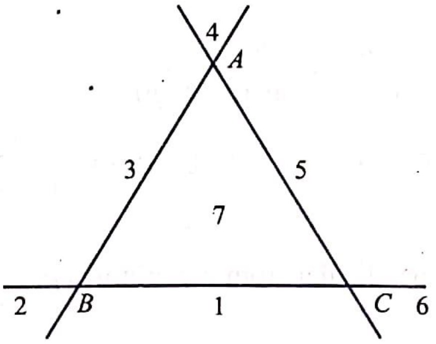


Fig. 7.35

$$x + 7y + 11 = 0,\ x - 3y + 1 = 0,\ 3x + y - 7 = 0$$

respectively. Now (0, 0) will be within the ΔABC if and only if (0, 0) and A are on the same side of BC; (0, 0) and B are on the same side of CA; (0, 0) and C are on the same side of AB.

(0, 0) and A are on the same side of BC whose equation is $x + 7y + 11 = 0$ since $2 + 7 + 11 = 20 > 0$ and $0 + 7(0) + 11 = 11 > 0$ are of the same sign.

Similarly, B and (0, 0) are on the same side of CA because $3 - 3(-2) + 1 = 10 > 0$ and $0 - 3(0) + 1 = 1 > 0$. Finally, substituting the coordinates of C in $3x + y - 7$ we get $3(-4) + (-1) - 7 = -20 < 0$; also $3(0) + (0) - 7 = -7 < 0$. Hence (0, 0) lies within ΔABC.

Alternately we can do the above example as follows.

Equation to OA is $y = x/2$ or $x - 2y = 0$.

Substituting the coordinats of $B(3, -2)$ and $C(-4, -1)$ in the equation to OA we see that $3 - 2(-2) = 7 > 0$ and $-4 - 2(-1) = -2 < 0$, So B and C are on the opposite sides of OA. This means that $O(0, 0)$ lies within either the region 1, 4 or 7 (Fig. 7.35). Similarly C and A are on the opposite sides of OB. Again this means that O lies within the region either 2, 5 or 7. Hence O (0, 0) lies within the region 7; *i.e.*, O lies with ΔABC.

To find the length of the perpendicular from the origin to the straight line $ax + by + c = 0$.

If ON is the perpendicular from O on $ax + by + c = 0$ then the straight line ON may be put in the normal form $x \cos \alpha + y \sin \alpha = p$ where $p = |ON|$ (Fig. 7.36).

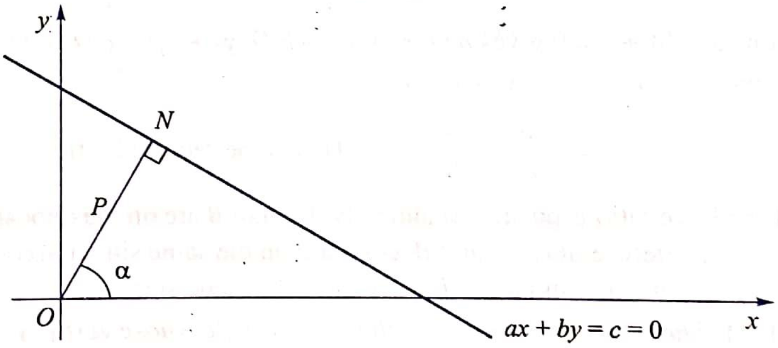


Fig. 7.36

Now $ax + by + c = 0$ and $x \cos \alpha + y \sin \alpha = p$ represent the same straight line implies that

$$\frac{a}{\cos \alpha} = \frac{b}{\sin \alpha} = \frac{c}{(-p)}.$$

Eliminating α using $\cos^2 \alpha + \sin^2 \alpha = 1$ we get $p = \pm \dfrac{c}{\sqrt{a^2 + b^2}}$.

Thus the length of the perpendicular from the origin is $p = \left| \dfrac{c}{\sqrt{a^2 + b^2}} \right|$

Note. If $c < 0$ in $ax + by + c = 0$, then $p = \dfrac{-c}{\sqrt{a^2 + b^2}}$ and if $c > 0$ then $p = \dfrac{c}{\sqrt{a^2 + b^2}}$.

SHIFTING OF ORIGIN

Suppose we fix a Cartesian frame of reference OX, OY and A is a point with coordinates (x_1, y_1). Consider a new pair of axes AX', AY' through A parallel to the original axes OX, OY respectively (Fig. 7.37).

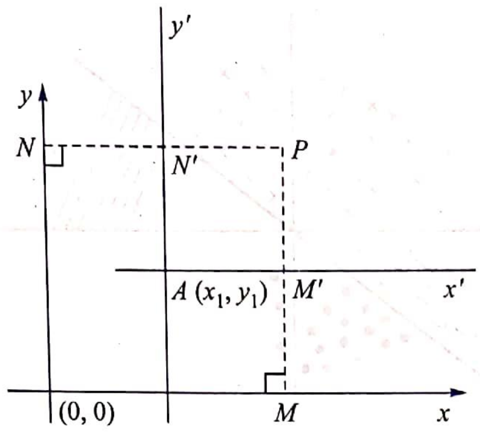


Fig. 7.37

With respect to the new axes, whose origin now is A, the coordinates of A are (0, 0). Let P be any point in the plane whose coordinates are $P(x, y)$ and $P(x', y')$ with respect to OX, OY and AX', AY' axes respectively. Then we have $OM = x$, $AM' = x'$, $ON = y$, $AN' = y'$ (see Fig. 7.37 for the explanation of the symbols M, M' etc. We observe that $OM = x = x_1 + x'$ and $ON = y = y_1 + y'$. Thus the transformation equations from the old set of axes to the new set of axes are $x' = x - x_1$, $y' = y - y_1$.

For example, if we shift the origin to (– 1, 3) with the axes remaining parallel, the transformation equations are $x' = x - (-1) = x + 1$, $y' = y - 3$. This means that if P is (3, – 2) with respect to the old axes, then P is (3 + 1, – 2 – 3) = (4, – 5) with respect to the new axes.

To find the perpendicular distance of $A(x_1, y_1)$ from $ax + by + c = 0$:

Shifting the origin to $A(x_1, y_1)$ with the axes remaining parallel, the equation to the given line becomes $a(x' + x_1) + b(y' + y_1) + c = 0$ or $ax' + by' + (ax_1 + by_1 + c) = 0$, with respect to the new axes. A is the origin of the new axes and hence the perpendicular distance of A from $ax' + by' + (ax_1 + by_1 + c) = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

EXAMPLE 12. *Find the locus of a point which moves such that the sum of the perpendicular distances from it on two given straight lines is a constant.*

SOLUTION. We may take one of the two given straight lines to be our x-axis and the point of intersection of the given lines as our origin. Observe that we have the freedom of choosing our axes, depending on the problem. (Reader, this is where we intelligently exploit the convenience of coordinate geometry). By our choice of the axes one of the given lines is $y = 0$ and let the other be $mx - y = 0$. If $P(x', y')$ is the variable point, its distances

from the two given lines are $|y'|$ and $\dfrac{|mx' - y'|}{\sqrt{1 + m^2}}$

We are given that P moves such that $|y'| + \dfrac{|mx' - y'|}{\sqrt{1 + m^2}} = k =$ constant.

$\therefore$ The locus of p is $\sqrt{1 + m^2}\,|y| + |mx - y| = k\sqrt{1 + m^2}$.

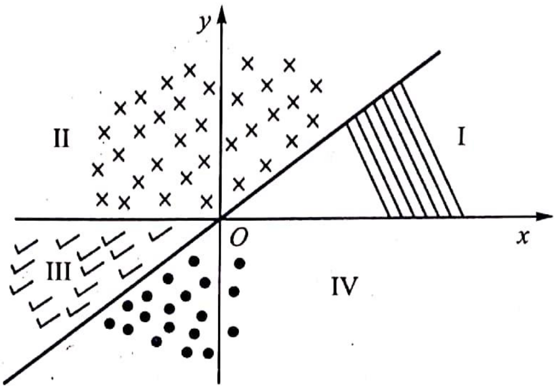


Fig. 7.38

In region I where $y > 0$, $mx > y$ the locus is the straight line $mx + (\sqrt{1+m^2}-1)y - k\sqrt{1+m^2} = 0$. In region II where $y > 0$, $mx < y$ the locus is the straight line $(1 + \sqrt{1+m^2})y - mx = k\sqrt{1+m^2}$. In region III, where $y < 0$, $mx < y$ the locus is the straight line $(1 - (1 - \sqrt{1+m^2})y - mx = k\sqrt{1+m^2}$. In region IV, the locus is $mx - (1 + \sqrt{1+m^2})y = k\sqrt{1+m^2}$.

EXAMPLE 13. *Find the incentre of the triangle whose sides have the equations*

$$x + y - 7 = 0, \quad x - y + 1 = 0 \quad and \quad x - 3y + 5 = 0.$$

SOLUTION. Let I (x_1, y_1) be the incentre of Δ ABC whose sides have the given equations. Then the perpendicular distance of I from BC is $= \frac{|x_1 + y_1 - 7|}{\sqrt{2}}$.

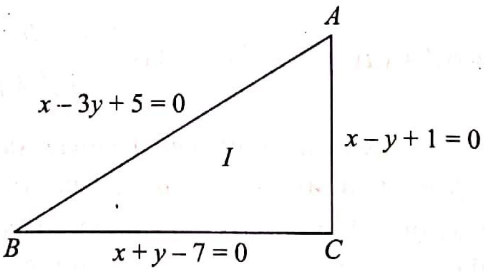


Fig. 7.39

From Fig. 7.39, it is clear that (0, 0) and I are on the opposite sides of BC. Hence $x_1 + y_1 - 7 > 0$.

$\therefore$ $$r = + \frac{x_1 + y_1 - 7}{\sqrt{2}}.$$ Similarly the distance of I from CA is

$$= \frac{|x_1 + y_1 + 1|}{\sqrt{2}} = -\frac{(x_1 - y_1 + 1)}{\sqrt{2}}$$

since O and I are on the opposite sides of $x - y + 1 = 0$. The distance of I from AB is

$$r = \frac{|x_1 - 3y_1 + 5|}{\sqrt{1+9}} = \frac{x_1 - 3y_1 + 5}{\sqrt{10}}$$

since O and I are on the same side of AB (Fig. 7.39).

Thus $$r = \frac{x_1 + y_1 - 7}{\sqrt{2}} = -\frac{(x_1 - y_1 + 1)}{\sqrt{2}} = \frac{x_1 - 3y_1 + 5}{\sqrt{10}}.$$

which leads to $x_1 = 3, y_1 = 1 + \sqrt{5}$.

∴ The incentre is $(3, 1 + \sqrt{5})$.

EXAMPLE 14. *Prove that the area of a parallelogram is $p_1p_2/\sin\alpha$ where p_1, and p_2 are the distances between the parallel sides and α is any angle of the parallelogram. Hence prove that the area of the parallelogram formed by $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_2x + b_2y + d_2 = 0$ is*

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{(a_1b_2 - a_2b_1)}\right|.$$

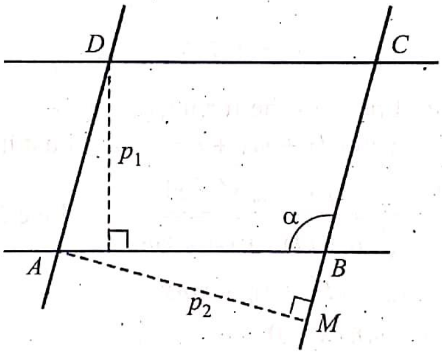


Fig. 7.40

SOLUTION. We have, area of the parallelogram $ABCD = AB \cdot p_1 \cdot p_1$ (From ΔABM) $= (p_2/\sin(180 - \alpha))$

Thus area of parallelogram $ABCD = (p_1 p_2)/\sin\alpha$.

Let AB be $a_1x + b_1y + c_1 = 0$, BC be $a_2x + b_2y + c_2 = 0$, CA be $a_1x + b_1y + d_1 = 0$, and DA be $a_2x + b_2y + d_2 = 0$. Then the distance between the parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + d_1 = 0$ is given by

$$p_1 = \frac{|c_1 - d_1|}{\sqrt{a_1^2 + b_1^2}}. \text{ Similarly } p_2 = \frac{|c_2 - d_2|}{\sqrt{a_2^2 + b_2^2}}$$

The acute angle between AB and BC is given by $\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$ where m_1, m_2 are the slopes of AB, BC.

$$\therefore \quad \tan^2\alpha = \frac{\left(-\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)^2}{\left(1 + \frac{a_1a_2}{b_1b_2}\right)^2} = \frac{(a_2b_1 - a_1b_2)^2}{(b_1b_2 + a_1a_2)^2}.$$

$$\therefore \quad \sin^2 \alpha = \frac{1}{\left(\frac{a_1a_2+b_1b_2}{a_2b_1-a_1b_2}\right)^2+1} = \frac{(a_2b_1-a_1b_2)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)},$$

$$\therefore \quad \text{Area of parallelogram } ABCD = \frac{p_1p_2}{\sin\alpha}$$

$$= \frac{|c_1-d_1||c_2-d_2|}{|a_1b_2-a_2b_1|}$$

EXAMPLE 15. *The straight line $lx + my + n = 0$ bisects an angle between a pair of lines of which $px + qy + r = 0$ is one. Find the other line.*

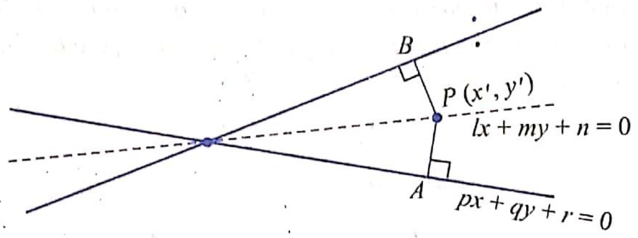


Fig. 7.41

SOLUTION. The required line is of the form $(px + qy + r) + k\,(lx + my + n) = 0$. For any point $P(x', y')$ on the bisector $lx + my + n = 0$, we must have $PA = PB$ (Fig. 7.41).

$$\therefore \quad \frac{|px'+qy'+r|}{\sqrt{p^2+q^2}} = \frac{|px'+qy'+r|}{\sqrt{(p+kl)^2+(q+km)^2}} \quad (\text{Since } lx'+my'+n=0)$$

This implies that $p^2 + q^2 = (p + kl)^2 + (q + km)^2$

$$\therefore \quad (l^2+m^2)\,k^2 + 2(pl+qm)\,k = 0$$

Hence $\qquad k = 0 \text{ or } k = -2\dfrac{(pl+qm)}{l^2+m^2}$

Here $k \neq 0$ (Why?) and hence $k = -2\dfrac{(pl+qm)}{l^2+m^2}$

This gives the required line as

$$px + qy + r - 2\frac{(pl+qm)}{l^2+m^2}(lx+my+n) = 0$$

or $\qquad (l^2 + m^2)(px + qy + r) - 2(pl + qm)(lx + my + n) = 0.$

Remark. Consider two straight lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$. Any point on a bisector (internal or external) of the angle between this pair of lines is equidistant from the two lines. Hence the bisectors have the equation

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

EXAMPLE 16. *Show that the internal bisectors of the angles of a triangle are concurrent.*

SOLUTION. Let the three sides be $l_j \equiv a_jx + b_jy + c_j = 0, j = 1, 2, 3$. There is no loss in generality in assuming that (0, 0) lies within the triangle. In fact, we may choose

any point inside the triangle as our origin and assume the sides to be $l_j \equiv a_j x + b_j y + c_j = 0, j \equiv 1,2,3$. We may also assume that $c_j > 0$ for all j.

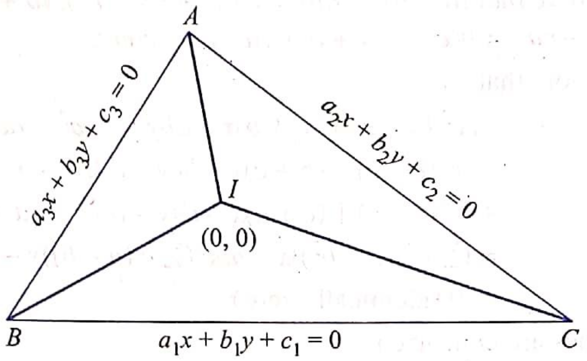


Fig. 7.42

For any point P on the internal bisector of $\angle A$, lying within ΔABC, the origin and P are on the same side of AB, AC. Hence if P is (x', y') we have either $a_2x' + b_2y' + c_2$ and $a_3x' + b_3y' + c_3$ both positive or both negative (note that by our choice c_1, c_2, c_3 are all positive). Thus the internal bisector of $\angle A$ is

$$\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} = \frac{a_3x + b_3y + c_3}{\sqrt{a_3^2 + b_3^2}}$$

Similarly the other internal bisectors are $\dfrac{a_3x + b_3y + c_3}{\sqrt{a_3^2 + b_3^2}} = \dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}$,

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

In other words, the internal bisectors are $\dfrac{l_1}{\sqrt{a_1^2 + b_1^2}} - \dfrac{l_2}{\sqrt{a_2^2 + b_2^2}} = 0$

$$\frac{l_2}{\sqrt{a_2^2 + b_2^2}} - \frac{l_3}{\sqrt{a_3^2 + b_3^2}} = 0, \quad \frac{l_3}{\sqrt{a_3^2 + b_3^2}} - \frac{l_2}{\sqrt{a_1^2 + b_1^2}} = 0$$

where $l_j \equiv a_j x + b_j y + c_j$ for $j = 1, 2, 3$. Now adding the three equations we get

$$\Sigma \frac{l_1}{\sqrt{a_1^2 + b_1^2}} - \frac{l_2}{\sqrt{a_2^2 + b_2^2}} \equiv 0$$

and hence the three internal bisectors are concurrent.

EXAMPLE 17. *Find the value of a for which the three lines $2x + y - 1 = 0$, $ax + 3y - 3 = 0$, $3x + 2y - 2 = 0$ are concurrent.*

SOLUTION. Solving $2x + y - 1 = 0$ and $3x + 2y - 2 = 0$, we get the point $(0,1)$ as the point of intersection. Now, whatever be 'a' $(0,1)$ always lies on $ax + 3y - 3 = 0$. Hence the three lines are concurrent for all values of a.

EXAMPLE 18. *If the lines $x + 2y = 9$, $3x - 5y = 5$, $ax + by = 1$ are concurrent then prove that $5x + 2y = 1$ passes through (a, b).*

Solving $x + 2y = 9$ and $3x - 5y = 5$ we get the point (5, 2). If $ax + by = 1$ passes through (5,2) then $5a + 2b = 1$. This means that $5x + 2y = 1$ passes through (a, b).

EXAMPLE 19. *Prove that $(a + b)x - aby = c(a^2 + ab + b^2)$, $(b + c)x - bcy = a(b^2 + bc + c^2)$, $(c + a)x - cay = b(c^2 + ca + a^2)$ are concurrent.*

SOLUTION. We note that

$$\begin{aligned} & c^2(a-b)\{(a+b)x - aby - c(a^2+ab+b^2)\} \\ & + a^2(b-c)\{(b+c)x - bcy - a(b^2+bc+c^2)\} \\ & + b^2(c-a)\{(c+a)x - cay - b(c^2+ca+a^2)\} \\ & = (\Sigma c^2(a^2-b^2))x - abc(\Sigma c(a-b))y - \Sigma c^3(a^3-b^3) \\ & \equiv 0 \text{ (identically zero).} \end{aligned}$$

Hence the three lines are concurrent.

EXAMPLE 20. *Find the equations of the diagonals formed by the lines $2x - y + 7 = 0$, $2x - y - 5 = 0$, $3x + 2y - 5 = 0$ and $3x + 2y + 4 = 0$.*

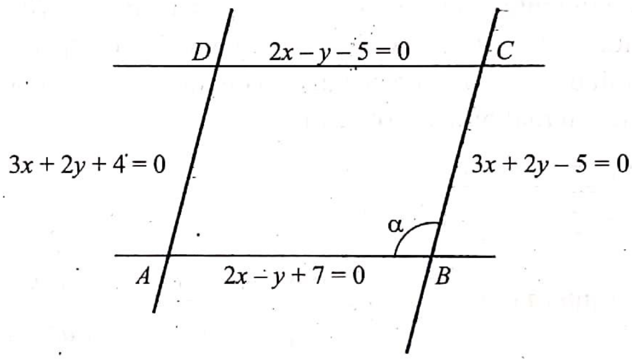


Fig. 7.43

SOLUTION. Equation to AC must be of both the forms

$$2x - y + 7 + \lambda(3x + 2y + 4) = 0 \text{ and}$$
$$2x - y - 5 + \mu(3x + 2y - 5) = 0.$$

This gives
$$\frac{2+3\lambda}{2+3\mu} = \frac{-1+2\lambda}{-1+2\mu} = \frac{7+4\lambda}{-5-5\mu}$$

$$\frac{2+3\lambda}{2+3\mu} = \frac{-1+2\lambda}{-1+2\mu} \text{ gives } \lambda = \mu$$

$\therefore$ $$7 + 4\lambda = -5 - 5\lambda \text{ or } \lambda = -(4/3)$$

Hence equation to AC is

$$2x - y + 7 - (4/3)(3x + 2y + 4) = 0 \text{ or}$$
$$6x + 11y - 5 = 0$$

Similarly BD is of both the forms

$$2x - y + 7 + k(3x + 2y - 5) = 0$$
$$2x - y - 5 + l(3x + 2y + 4) = 0.$$

This gives
$$\frac{2+3k}{2+3l} = \frac{-1+2k}{-1+2l} = \frac{7-5k}{-5+4l}.$$

Again, these equations give $k = 1$ and $7 - 5k = -5 + 4k$ or $k = (4/3)$.

Equation to BD is $2x - y + 7 + (4/3)(3x + 2y - 5) = 0$

or $$18x + 5y + 1 = 0.$$

Thus the diagonals have the equations $6x + 11y - 5 = 0$, $18x + 5y + 1 = 0$.

EXAMPLE 21. *A straight line moves so that the sum of the reciprocals of its intercepts on the coordinate axes is constant. Show that it passes through a fixed point.*

SOLUTION. Let the variable line be $\frac{x}{a} + \frac{y}{b} = 1$.

Then we are given that $\frac{1}{a} + \frac{1}{b} = K = a$ constant.

Therefore, the variable line takes the form

$$\frac{x}{a} + \left(k - \frac{1}{a}\right)y - 1 = 0 \quad \text{or}$$

$\frac{1}{a}(x - y) + (K y - 1) = 0$. This represents a straight line a through the intersection of $x - y = 0$ and $Ky - 1 = 0$. They intersect at $(1/K, 1/K)$ and hence $\frac{x}{a} + \frac{y}{b} = 1$ always passes through the fixed point $\left(\frac{1}{K}, \frac{1}{K}\right)$.

EXAMPLE 22. *ABCD is a variable rectangle having its sides parallel to fixed directions. The vertices B and D lie on $x = a$ and $x = -a$ and A lies on the line $y = 0$. Find the locus of C.*

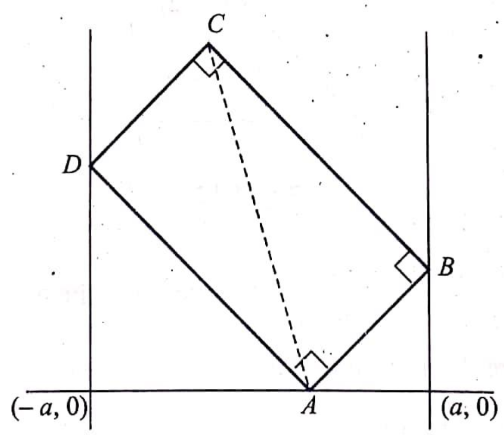


Fig. 7.44

SOLUTION. Let A be $(x_1, 0)$, B be (a, y_2) and D be $(-a, y_4)$. We are given AB and AD have fixed directions and hence their slopes are constants, say m_1 and m_2.

$$\therefore \qquad \frac{y_2}{a - x_1} = m_1, \text{ and } \frac{y_4}{-a - x_1} = m_2.$$

Further $m_1 m_2 = -1$ since $ABCD$ is a rectangle.

$$\frac{y_2}{a - x_1} = m_1 \text{ and } \frac{y_4}{-a - x_1} = -\frac{1}{m_1}.$$

The midpoint of BD is $\left(0, \frac{y_2 + y_4}{2}\right) = \left(\frac{x_1 + x}{2}, \frac{y}{2}\right)$

= midpoint of AC where C is taken to be (x, y).

This gives $x = -x_1$ and $y = y_2 + y_4$. So C is $(-x_1, y_2 + y_4)$.

Also $\frac{y_2}{a - x_1} = m_1$ and $\frac{y_4}{a + x_1} = +\frac{1}{m_1}$ gives

the locus of C as

$$(m_1^2 - 1)x + m_1 y = (m_1^2 + 1).$$

EXAMPLE 23. *Each side of a square is of length 6 units and the centre of the square is (– 1, 2). One of its diagonals is parallel to x + y = 0. Find the coordinates of the vertices of the square.*

SOLUTION. Let $ABCD$ be the given square with centre $(-1, 2)$ and side of length 6. BD is parallel to $x + y = 0$. Equation to BD is $x + y = 1$. Hence the equation to AC is $x - y + 3 = 0$ (note that $AC \perp BD$).

We have $|OC| = |OB| = |OA| = |OD| = 3\sqrt{2}$ units.

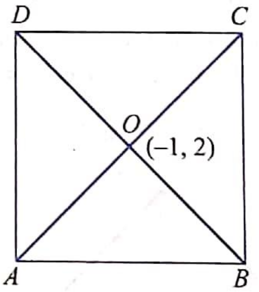


Fig. 7.45

We may write AC as $\frac{x - (-1)}{\cos 45°} = \frac{y - 2}{\sin 45°} = r$ where r is the algebraic distance of (x, y) from $(-1, 2)$. Therefore A and C are given by

$$\frac{x+1}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm 3\sqrt{2}$$ or A is $(2, 5)$ and C is $(-4, -1)$. Again, we may

write BD as $$\frac{x+1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = r.$$

B and D are given by $$\frac{x+1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm 3\sqrt{2}$$

B is $(-4, 5)$ and D is $(2, -1)$.

The vertices of the square are $(2, 5)$, $(-4, 5)$, $(-4, -1)$ and $(2, -1)$.

EXERCISE 7.2

1. Find the slopes and the intercepts upon the axes of the following lines and reduce each to normal form.
   (i) $3x - 4y + 12 = 0$ (ii) $12x + 5y = 39$
   (iii) $15x - 8y + 34 = 0$ (iv) $11x + 60y = 61$
   (v) $x - y = 8$.
2. Write the following straight lines in the parametric form
   $$(x - x_1)/\cos\theta = (y - y_1)/\sin\theta = r.$$
   (i) through (2, 3) with slope 2
   (ii) through (1, 4) with slope – 1/3
   (iii) through (1, 3) and (4, 2).
3. Find the equations to the straight lines which join the origin and the points of trisection of the portion of the line $x + 3y - 12 = 0$ intercepted between the coordinate axes.
4. If $(x_1, y_1,)$ is the midpoint of the portion of a straight line intercepted between the coordinate axes, prove that the equation of the line is $x/2x_1 + y/2y_1 = 1$.
5. Find the equations of the straight lines through the origin whose intercepts between the lines $5x + 12y = 15$ and $5x + 12y = 30$ are each equal to three.
6. Show that the lines $3x + y + 4 = 0$, $3x + 4y - 15 = 0$ and $24x - 7y - 3 = 0$ form an isosceles triangle.
7. Find the area of the triangle formed by the lines $2x - y + 4 = 0$, $3x + 2y - 5 = 0$, $x + y + 1 = 0$.
8. Straight lines are drawn from A (3, 2) to meet the line $6x + 7y = 30$ and these straight lines are bisected. Find the locus of the midpoints.
9. Find the area of the triangle formed by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$.
10. Find the acute angle between $3x - 2y + 3 = 0$ and $2x + y - 5 = 0$.
11. Find the equations of the lines through (2, 3) which make 45° with $3x - y + 5 = 0$.
12. If (h, k) is the foot of the perpendicular from (x_1, y_1) to the straight line $lx + my + n = 0$, show that $(h - x_1)/l = (k - y_1)/m\ (lx_1 + my_1 + n)/\ (l^2 + m^2)$.
13. A vertex of an equilateral triangle is at (2, 3) and the opposite side is $x + y = 2$. Find the equations to the other sides of the triangle.
14. A triangle is formed by the lines $ax + by + c = 0$, $lx + my + n = 0$, $px + qy + r = 0$. Show that $(ax + by + c) / (ap + bq) = (lx + my + n) / (lp + mq)$ passes through the orthocentre.
15. Show that the origin is within the triangle formed by the lines $4x + 7y + 19 = 0$, $4x + y - 11 = 0$ and $4x - 5y + 7 = 0$.
16. Find the in radius of the triangle formed by the lines $x = 0$, $y = 0$ and $x/3 + y/4 = 1$.
17. Show that the following pair of equations represents the same family of straight lines $2x + 3y - 8 + \lambda(4x - 7y + 10) = 0$ and $3x + 4y - 11 + v(2x - 5y + 8) = 0$.
18. Write down the equations of the bisectors of the angles between the lines.
    (i) $x + 2y + 3 = 0$ and $2x - y - 5 = 0$ (ii) $4x + 3y + 10 = 0$ and $12x - 5y + 2 = 0$.
19. Prove that $x + y + 2 = 0$, $x - 7y = 2$, and $6x + 8y + 13 = 0$, $2y + 1 = 0$ have the same angular bisectors.
20. Find the directions in which a straight line must be drawn through (1, 2) so that its point of intersection with $x + y = 4$ may be at a distance $(1/3)\sqrt{6}$ from the point.
21. Show that $2x - 3y + 5 = 0$, $3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$ are concurrent.

-

Miscellaneous Problems page 518

MISCELLANEOUS PROBLEMS

1. Let α and β be the roots of

$$z + \frac{1}{z} = 2\cos(\phi + i\sin\phi) \qquad 0 < \phi < \pi$$

(*a*) Show that $\alpha + i$ and $\beta + i$ have the same argument and $|\alpha - i| = |\beta - i|$.

(*b*) Find the locus of the roots α and β in **C** as φ varies from 0 to 2π.

2. Suppose α, β, γ and δ are real numbers such that

$$\alpha + \beta + \gamma + \delta = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 = 0$$

Prove that

$$\alpha(\alpha + \beta)(\alpha + \gamma)(\alpha + \delta) = 0.$$

3. Let α and β be the zeros of $x^2 - 6x + 1 = 0$. Show that $\alpha^n + \beta^n$ is an integer for any n and it is not divisible by 5.

4. A square matrix A is said to be *orthogonal* if $AA^T = I$. Show that the following matrices are orthogonal:

(*a*) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (*b*) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*c*) The $n \times n$ matrix obtained by interchanging any two rows of the identity matrix I_n.

5. Show that, if

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

then $$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

6. Find all polynomials $p(x)$ such that

$$p(x^2) = \{p(x)\}^2.$$

7. Let a_1, a_2, a_3, a_4 and b be real numbers such that

$$b + \sum_{k=1}^{4} a_k = 8; \; b^2 + \sum_{k=1}^{4} a_k^2 = 16$$

Find the maximum value of b.

8. For any natural number n prove that

$$2\sqrt{n}-2<\sum_{k=1}^{n}\frac{1}{\sqrt{K}}<2\sqrt{n}-1.$$

9. Factorize:

(a) $\Sigma(bc-a^2)(ca-b^2)$ (b) $\Sigma(bc-a^2)^2(ca-b^2)^2$.

10. If $\tan A$ and $\tan B$ are the roots of $ax^2+bx+c=0$, evaluate

$$a\sin^2(A+B)+b\sin(A+B)\cos(A+B)+c\cos^2(A+B).$$

11. Show that there exists a convex hexagon in the plane such that

(i) all its interior angles are equal;

(ii) the lengths of its sides are 1, 2, 3, 4, 5, 6 in some order.

12. Find the remainder when 19^{92} is divided by 92.

13. Determine all pairs (m, n) of positive integers m, n for which 2^m+3^n is a perfect square.

14. Find the number of positive integers $n \le 1991$ such that 6 is a factor of n^2+3n+2.

15. Let a, b, c be three real numbers with $0<a<1$, $0<b<1$, $0<c<1$ and $a+b+c=2$. Prove that

$$\frac{a}{1-a}\,\frac{b}{1-b}\,\frac{c}{1-c}\ge 8.$$

16. Solve for real numbers x, y, z:

$$x+y-z=4$$
$$x^2-y^2+z^2=-4$$
$$xyz=6.$$

17. Six generals propose locking a safe containing top secret with a number of different locks. Each general will be given keys to certain of these locks. How many locks are required and how many keys must each general have so that, unless at least four generals are present, the safe cannot be opened?

18. For any positive integer n, let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$ (For instance, $s(2)=3$). Determine the set of positive integers n for which $s(n)=5$.

19. If

$$D_n=\begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & a_2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & a_3 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & a_n \end{vmatrix}$$

Show that $D_n=a_nD_{n-1}+D_{n-2}$.

20. There are ten objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the ten objects can be divided into two groups that balance each other in weight.

21. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of real numbers such that either both are increasing or both are decreasing. Prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le n\sum_{k=1}^{n} a_k b_k.$$

(This is known as Chebyshev's inequality).

22. (*a*) If $p = (x_1, y_1)$ and $Q = (x_2, y_2)$ with O as origin and $OPRQ$ is the ▱gm with OP, OQ as adjacent sides, then show that

Area of the ▱gm $OPRQ$ is $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

(*b*) If $A = (x_0, y_0)$, $P = (x_1, y_1)$ & $Q = (x_2, y_2)$ then show that the area of the parallelogram with adjacent sides AP, AQ is

$$\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}$$

(*c*) Generalise this result to three dimensions.

23. Find all integers n such that

$$nx^2 + (n+1)x + (n+2) = 0$$

has only rational roots.

24. Find all integers n and a such that $t^2 - t + 1$ divides $t^n + t + a$.

25. Show that there is no polynomial $p(x)$ in $Z[x]$ for which $p(k)$ is a prime for all integers $k \ge 0$.

26. Let $A = [1, 2, 3, \ldots 100\}$ and B be a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

27. If a, b, c, d are four nonnegative real numbers and $a + b + c + d = 1$. Show that $ab + bc + cd \le 1/4$.

28. Given a triangle ABC, define

$$x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$$

$$y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$$

$$z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$$

Prove that $x + y + z + xyz = 0$.

29. Show that there do not exist four distinct real numbers a, b, c, d such that

$$a^3 + b^3 = c^3 + d^3$$

and $\quad a + b = c + d.$

30. Determine the largest number in the infinite sequence:

$$1^2, \sqrt[2]{2}, \sqrt[3]{3}, \ldots \sqrt[n]{n}$$

31. Prove that if a, b, c are odd integers then $ax^2 + bx + c = 0$ cannot have a rational number as a root.

32. Let P be any point inside a triangle ABC. If AP, BP and CP meet the sides BC, CA, AB at D, E, F respectively. Prove that

$$PD + PE + PF < \max(a, b, c).$$

33. Construct a quadrilateral which is not a ▱gm and in which a pair of opposite angles and a pair of opposite sides are equal.

34. Find all polynomials $p(x)$ such that

$$xp(x-1) = (x-15)p(x).$$

35. Find all natural numbers n for which the product of its digits is $n^2 - 13n - 25$.

36. Let $\Delta(x, y)$ be the numerical area of a triangle whose vertices are $(0, 0)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Show that

$$\Delta(x, y) = \Delta(y, x);$$
$$\Delta(\alpha x, y) = |\alpha| \Delta(x, y);$$
$$\Delta(x + \alpha y, y) = \Delta(x, y).$$

37. With the notation of Qn. 36 above,

(*a*) If $x = (1, 2)$, $y = (-1, 4)$, $z = (1, -3)$ show that $\Delta(x + y, z) = \Delta(x, z) + \Delta(y, z)$

(*b*) If $x = (2, 4)$, $y = (2, 1)$, $z = (3, 5)$ show that $\Delta(x + y, z) = -\Delta(x, z) + \Delta(y, z)$;

(*c*) If $x = (2, 5)$, $y = (-1, 2)$, $z = (-1, 3)$ show that $\Delta(x + y, z) = \Delta(x, z) - \Delta(y, z)$.

38. If $\Delta'(x, y)$ be defined as the (signed) area of the triangle whose vertices are $(0, 0)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ show that

$$\Delta'(x + y, z) = \Delta'(x, z) + \Delta'(y, z)$$
$$\Delta'(x, y) = -\Delta'(y, x)$$
$$\Delta'(\alpha x, y) = \alpha\Delta'(x, y)$$

and $$\Delta'(x + \alpha y, y) = \Delta'(x, y)$$

Calculate $\Delta'(x + y, z)$ in all three cases (*a*), (*b*) and (*c*) of Qn. 37 and contrast with the behaviour of $\Delta(x + y, z)$. Can you explain why $\Delta(x + y, z)$ has three different expressions in (*a*) (*b*) (*c*) of Qn. 37?

39. Show that

$$\Sigma \sin^3 \alpha \sin(\beta - \gamma) = -\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \beta) \times \sin(\alpha + \beta + \gamma).$$

40. Find the value of the positive integer n for which the equation

$$\Sigma_{i=1}^{n}(x + i - 1)(x + i) = 10n.$$

41. Determine the set of all positive integers n for which 3^{n+1} divides $2^{2^n} + 1$.

42. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

43. Show that if R and r are the circumradius and the in-radius of a nonobtuse-angled triangle, and h be the greatest altitude then

$$R + r \leq h.$$

44. A real number α is said to be algebraic if α is a zero of a polynomial in $Z[x]$, *e.g.*, $\sqrt{2}$ is algebraic since it satisfies $x^2 - 2 = 0$. If α and β are algebraic show that $\alpha + \beta$ and $\alpha\beta$ are algebraic.

45. Eliminate x, y from the equations:

(a) $x^2 + xy = a^2$
$x^3 + xy = b^2$
$x^2 + y^2 = c^2$

(b) $(b - x)(c - y) = a^2$
$(c - x)(a - y) = b^2$
$(a - x)(b - y) = c^2$

(c) $\dfrac{1}{x-a} + \dfrac{1}{y-a} = \dfrac{1}{b}$
$\dfrac{1}{x-b} + \dfrac{1}{y-b} = \dfrac{1}{b}$
$x^2 + y^2 = 2(a^2 + b^2)$

(d) $x^2 - y^2 = px - qy$
$4xy = py + qx$
$x^2 + y^2 = 1$

(e) $4(x^2 + y^2) = ax + by$
$2(x^2 - y^2) = ax - by$
$xy = c^2$

(f) $ax^2 + by^2 = ax + by = \dfrac{xy}{x+y} = c$

(g) $\dfrac{x}{a^2 - y^2} + \dfrac{y}{a^2 - x^2} = \dfrac{1}{b}$
$xy = c^2 \; (a \neq 0)$

(h) $x + y = a$
$x^2 + y^2 = b^2$
$x^3 + y^3 = c^3$

46. Consider the collection of all three-element subsets drawn from the set $\{1, 2, 3, ..., 300\}$. Determine the number of those subsets for which the sum of the elements is a multiple of 3.

47. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the common point O are collinear.

48. Find all possible values of x, y, z such that $AA^T = I$ where

$$A = \begin{pmatrix} 1/\sqrt{2} & 2/3 & x \\ 1/\sqrt{2} & -2/3 & y \\ 0 & 1/3 & z \end{pmatrix}.$$

49. If
$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

Calculate $\lambda I - A$ and solve $|\lambda I - A| = 0$ for λ. Check that $A^3 - 4A + 3I$ is the zero matrix.

50. Solve for x:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-x & 1 & \cdots & 1 \\ 1 & 1 & 2-x & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & n-x \end{vmatrix} = 0$$

51. Without expanding, prove that

(a) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(b) $$\begin{vmatrix} 2yz-x^2 & z^2 & y^2 \\ z^2 & 2zx-y^2 & x^2 \\ y^2 & x^2 & 2xy-z^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$$

(c) $$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}^2 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where A_{ij} is the cofactor of α_{ij} in the matrix (α_{ij}).

52. Consider the system

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

where, the coefficients satisfy:

(*i*) a_1, b_2, c_3 are all > 0

(*ii*) The other coefficients are all < 0 and

(*iii*) in each equation the sum of the coefficients is positive.

Prove that the system has only one solution, *viz*, $x = 0 = y = z$.

53. On a given circle, six points A, B, C, D, E and F are chosen at random independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, *i.e.*, have no common points.

54. A teacher distributes 7 books to 7 children (each child one book). On the next day she collects the books back & redistributes them in such a way that every child gets a new book. How many options are available for the teacher to do this?

55. If θ is expressed in radians, prove that

$$\cos(\sin \theta) > \sin(\cos \theta).$$

56. Find the number of permutations $(a_1, a_2, a_3, a_4, a_5, a_6)$ of $(1, 2, 3, 4, 5, 6)$ such that for any k, $1 \le k \le 5$, $(a_1, a_2, \dots a_k)$ is not a permutation of $1, 2 \dots, k$. (*e.g.*) $a_1 \ne 1$, (a_1, a_2) is not a permutation of $(1, 2)$ etc.

57. Let $A_1, A_2 \dots A_n$ be a regular polygon of n sides. If

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

determine n.

58. Find the number of ways in which one can place the numbers $1, 2, 3, \dots, n^2$ on the n^2 squares of an $n \times n$ chessboard, one on each, such that the numbers in each row and column are in $A.P.$

59. Eliminate x, y, z from the equations:

(a) $\frac{x}{a}(y+z-x) = \frac{y}{b}(z+x-y) = \frac{z}{c}(x+y-z)$;

$(ax + by + cz) = 0$.

(b) $x + y + z = a$ (c) $x + y - z = a$

$x^2 + y^2 + z^2 = b$ $x^2 + y^2 - z^2 = b^2$

$x^3 + y^3 + z^3 = c$ $x^3 + y^3 - z^3 = c^3$

$xyz = d$ $xyz = d^3$

(d) $x^2 - ayz = y^2 - bzx = z^2 - cxy$

$$= \frac{1}{2}(x^2 + y^2 + z^2)\ (x, y, z) \neq (0, 0, 0)$$

(e) $y^2 + z^2 - x(y + z) = a$ (f) $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = a$

$z^2 + x^2 - y(z + x) = b$ $\frac{x}{z} + \frac{y}{x} + \frac{z}{y} = b$

$x^2 + y^2 - z(x + y) = c$

$x^3 + y^3 + z^3 - 3xyz = d$ $\left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = c$

(g) x, y, z are the roots of $t^3 - at^2 + bt - c = 0$ and are also the sides of a right-angled triangle.

(h) $y + z = a(1 - yz)$

$z + x = b(1 - zx)$

$x + y = c(1 - xy)$

$x + y + z - xyz = d(1 - xy - yz - zx)$

(i) $\frac{y}{z} + \frac{z}{y} = a$ (j) $\frac{y}{z} - \frac{z}{y} = a$

$\frac{x}{z} + \frac{z}{x} = b$ $\frac{z}{x} - \frac{x}{z} = b$

$\frac{x}{y} + \frac{y}{x} = c$ $\frac{x}{y} - \frac{y}{x} = c$

60. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that

$$p(n)p(n + 1) = p(M).$$

61. Let f be a bijective (*i.e.*, one-one, onto) function from $A = \{1, 2, 3, ..., n)$ to itself. Show that there is a positive integer $M > 1$ such that

$$(\underbrace{f_o f_o f_o \cdots_o f}_{n \text{ times}})\ (i) = f(i)$$

for each $i \in A$.

62. Show that there is a natural number n such that $n!$ when written in decimal notation (*i.e.*, in base 10) ends exactly in 1993 zeros.

63. Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for five distinct integers a_1, a_2, a_3, a_4, a_5 we have

$$f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2.$$

Show that there does not exist an integer b such that $f(b) = 9$.

64. Determine all functions $f: R - \{0,1\} \to R$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

65. Determine all pairs (x, y) of nonnegative integers x and y for which $(xy - 7)^2 = x^2 + y^2$.

66. Let a, b, c, d be any four real numbers not all equal to zero. Prove that the roots of the polynomial

$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot all be real.

67. Let A denote a subset of

$$\{1, 11, 21, 31, ..., 541, 551\}$$

having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

68. Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

must be between the limits $\frac{3}{2}$ and 2. Can equality hold at either limit?

69. Let $m_1, m_2, m_3, ..., m_n$ be a rearrangement of the numbers $1, 2, ..., n$. Suppose that n is odd. Prove that the product

$$(m_1 - 1)(m_2 - 2) \ldots (m_n - n)$$

is an even integer.

70. Prove that the product of four consecutive natural numbers cannot be a perfect cube.

71. If a and b are positive real numbers and $a + b = 1$, prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq 12\frac{1}{2}.$$

72. Find the number of permutations of the n pairs of letters $a_1, a_1; a_2, a_2; a_3, a_3; \ldots a_n, a_n$ (each pair being identical), such that no two adjacent letters are identical.

73. Evaluate in 'closed' form.

(*a*) $1^3\binom{n}{1} + 2^3\binom{n}{2} + \ldots + n^3\binom{n}{n}$;

(*b*) $1^2 2\binom{n}{1} + 2^3 3\binom{n}{2} + \ldots + n^3(n+1)\binom{n}{n}$;

(*c*) If $S = \{1, 2, ..., n\}$ and $1 \leq r \leq n$

(*i*) $\sum\limits_{\substack{A \subset S \\ |A|=r}} (\min A)$ (*ii*) $\sum\limits_{\substack{A \subset S \\ |A|=r}} (\max A)$

(*iii*) $\sum\limits_{\substack{A \subset S \\ |A|=r}} (\min A)^2$ (*iv*) $\sum\limits_{\substack{A \subset S \\ |A|=r}} (\max A)^2$

74. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

75. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Show that, taking $f_0 = 1$ and $f_1 = 2$,

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2$$

Find the probability (in terms of f_n) that successive heads never appear when a coin is tossed n times, assuming that all possible outcomes of the n tosses are equally likely. What is the answer if $n = 8$?

76. A round-robin tournment is played amongst n-players. Each pair of players plays one game that culminates in a win for one player. A win earns 1 point; a loss 0 points. At the culmination of the tournament, player i has a_i points. $(0 \leq a_i \leq n-1)$. Suppose that $b = (b_1, b_2, ..., b_n)$ is a given n-tuple of nonnegative integers with $b_1 \leq b_2 \leq ... \leq b_n$. Prove that the n-tuple b could have arisen from a round-robin tournament as the set of final scores iff

(*i*) $\Sigma_{i=1}^{n} b_i = \binom{n}{2}$ (*ii*) $\Sigma_{i=1}^{k} b_i \geq \binom{k}{2}$ for each k, $1 \leq k \leq n$.

77. Let p be an interior point of a ΔABC. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30°.

78. The incircle of ΔABC touches BC at p. The line through p parallel to IA meets the incircle again at Q. The tangent to the incircle at Q meets AB, AC at C', B' respectively. Prove that $\Delta AB'C'$ is similar to ΔABC.

79. A permutation $a_1, a_2, ... a_n$ of $1, 2, ..., n$ is said to be *indecomposable* if n is the least positive integer j for which

$$\{a_1, a_2, ..., a_j\} = \{1, 2, ..., j\}$$

Let $f(n)$ be the number of indecomposable permutations of $1, 2, ..., n$. Find a recurrence relation for $f(n)$.

80. Show that the product of five consecutive positive integers is never a perfect square.

81. The incircle of ΔABC touches BC, CA at D, E respectively. Let BI meet DE at G. Show that AG is perpendicular to BG.

82. Let ABC be a triangle in plane Σ. Find the set of all points p (distinct from A, B, C) in the plane Σ such that the circumcircle of the triangles ABP, BCP, CAP have the same radius.

83. Prove that, if r be the inradius of ΔABC, the sum of the distances from a point p inside the triangle is at least $6r$.

84. Prove the following inequalities for a ΔABC:

(*a*) $3(bc + ca + ab) \leq (a + b + c)^2 < 4(bc + ca + ab)$

(*b*) $(a^2 + b^2 + c^2) \geq \frac{36}{35}\left(s^2 + \frac{abc}{s}\right)$

(*c*) $8(s-a)(s-b)(s-c) \leq abc$

(*d*) $abc < a^2(\Delta - a) + b^2(\Delta - b) + c^2(\Delta - c) \le \frac{3}{2}\, abc$

(*e*) $\Sigma bc(b + c) \ge 48(s - b)(s - c)(s - a)$

(*f*) $\frac{2s}{abc} \le \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

(*g*) $\frac{3}{2} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

(*h*) $0 < \sin A + \sin B + \sin C \le \frac{3}{2}\sqrt{3}$

(*i*) $\sin A + \sin B + \sin C \ge \sin 2A + \sin 2B + \sin 2C$

(*j*) $s \ge \frac{3}{2}\sqrt{6Rr}$.

85. Prove that a triangle is either acute, or right or obtuse–angled according as $a^2 + b^2 + c^2 - 8R^2$ is positive or zero or negative.

86. If $a^2 + b^2 > 5c^2$ in a ΔABC, show that c is the smallest side.

87. Let M be any point in the plane of a ΔABC. Find the minimum of $MA^2 + MB^2 + MC^2$.

88. Let S, I and O be the circumcentre, in-centre and the orthocentre of ΔABC. Prove that $SO \ge IO\sqrt{2}$.

89. If a polygon is inscribed in a circle and a second polygon is circumcircled by drawing tangents to the circle at the vertices of the first, prove that the product of the ⊥rs on the sides of the first polygon from a point of the circle, equals the product of the ⊥rs from the same point to the sides of the second polygon.

90. Prove that the sum of the ⊥rs from the vertices of a regular polygon of n sides to any line tangent to the circumscribed circle is equal to n times the radius.

91. For an acute angled triangle ABC prove that

$$s^2 \le \frac{27R^2}{27R^2 - 8r^2}(2R + r)^2$$

with the usual notation.

92. In a ΔABC prove that

$$\Sigma \cos\left(\frac{B-C}{2}\right) \le \frac{1}{\sqrt{3}}\left(\Sigma \sin A + \Sigma \cos\frac{A}{2}\right).$$

93. If $a, b > 0$ and $a + b = 1$, show that

$$(11 + (1/a^4))^{1/3} + (11 + (1/b^4))^{1/3} \ge 6.$$

94. In a triangle ABC, angle A is twice angle B. Show that $a^2 = b(b + c)$. Prove the converse.

95. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let T be the circumcentre of ΔAPB and H be the orthocentre of ΔCPD. Show that H, P, T are collinear.

96. Prove that if the Euler line of a triangle passes through a vertex then the triangle should either be a right angled triangle or on isosceles triangle or both.

97. If the Euler line of ΔABC is parallel to BC, Show that $\tan B \tan C = 3$.

98. Six different points are given on a circle. The orthocentre of the triangle formed by three of these points is joined to the centroid of the triangle formed by the other three points by a line segment. Prove that, when this is done for all possible triads of the six points chosen, the 20 line segments so formed are concurrent.

99. Prove that a straight line dividing the perimeter and the area, of the triangle in the same ratio, passes through the incentre.

100. Let P be the Fermat point of ΔABC. Prove that the Euler lines of Δs PAB, PBC, PCA are concurrent and the point of concurrence is G.

101. Prove that the three points on the circle whose pedal lines pass through the nine-point centre form an equilateral triangle.

102. Let A be one of the two points of two circles with centres X, Y respectively in the plane. The tangents at A to the two circles meet the circles again at B, C. Let point P be located so that $PXAY$ is a parallelogram. Show that P is also the circumcentre of ΔABC.

103. Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which $|AD| = \sqrt{|BD|.|CD|}$ where $|XY|$ denotes the distance between X and Y.

104. Let ABC be an acute angled triangle. For any point p lying within this triangle, let D, E, F denote the feet of the $\perp$rs from P onto the sides BC, CA, AB resply. Determine the set of all possible positions of the point P for which the ΔDEF is isosceles. For which position of P will ΔDEF be equilateral?

105. Given an angle QBP and a point L outside the angle, but in the same plane, show how to construct a straight line through L meeting BQ in A and BP in C such that ΔABC has a given perimeter.

106. Prove that the sum of the squares of the distances from any point on a circle to the vertices of a regular polygon of n sides incircled in the circle is constant and equals $2nR^2$.

107. Prove that the sum of the squares of all the connectors of the vertices of a regular polygon of n sides inscribed in a circle n^2R^2.

108. In a ΔABC the incircle Σ touches the sides BC, CA, AB at D, E, F resply. Let p be any point within Σ and let the segments AP, BP, CP meet Σ at X, Y, Z resply. Prove that DX, EY, FZ are concurrent.

109. Eliminate (x, y, z) from the equations:

(*a*) $ax^2 + by^2 + cz^2 = 0$
$ayz + bzx + cxy = 0$
$x^3 + y^3 + z^3 + \lambda xyz = 0$

(*b*) $(z + x - y)(x + y - z) = ayz$
$(x + y - z)(y + z - x) = bzx$
$(y + z - x)(z + x - y) = cxy$

(*c*) $y^2 + yz + z^2 = a$; $z^2 + zx + x^2 = b$;
$x^2 + xy + y^2 = c$; $xy + yz + zx = 0$.

(*d*) $a^2 + x^2 = 2\lambda ax$
$x^2 + y^2 = 2\lambda xy$
$y^2 + z^2 = 2\lambda yz$
$z^2 + b^2 = 2\lambda zb$
(a, c, y, z, b are all unequal)

(*e*) $ax^2 + by^2 + cz^2$
$= ax + by + cz = xy + yz + zx = 0$

(*f*) $x^2 + y^2 + z^2 = x + y + z = 1$
$\frac{a}{x}(x - p) = \frac{b}{y}(y - q) = \frac{c}{z}(z - r)$

(*g*) $lx + my + nz = mx + ny + lz$
$= nx + ly + mz = h(x^2 + y^2 + z^2) = 1$

(*h*) $x(x - a) = yz$; $y(y - b) = zx$;
$z(z - c) = xy$; $x^2 + y^2 + z^2 = d^2$.

ANSWERS TO SELECTED QUESTIONS IN THE EXERCISES AND PROBLEMS

EXERCISE 2.2. (p. 26)

1. (*a*) 3. (*b*) 9. (*c*) 333. (*d*) 6.

2. (*a*) 270 (*b*) $n(n+1)$ (*c*) 3702.

6. (*i*) $x = 9 + 22k$, $y = -11 - 21k$.

(*ii*) $x = 21 - 40k$, $y = 32 - 61k$.

(*iii*) $x = 21 - 4k$, $y = 21 - 5k$.

(*iv*) There exist solutions when $x = \ldots -6, -1, 4, 9 \ldots$. In particular, when $x = 4$, $y = 5 + 3k$, $z = -5 - 2k$.

10. 4 **13.** $a = b$ = any integer. **14.** φ.

15. $a = \pm 10, b = \pm 100; a = \pm 20, b = \pm 50;$
$a = \pm 100, b = \pm 10; a = \pm 50, b = \pm 20.$

17. 1 if a is even and 2 if a is odd. **21.** $(p+1)/2, (p-1)/2$.

EXERCISE 2.3. (p. 33)

2. $2^{15}3^{10}5^{6}$ **3.** 12 **4.** 16

6. 249 **7.** {1} **8.** 960

13. $\{252n \mid n \text{ is an odd integer}\}$.

PROBLEMS (Chapter 2) (p. 34)

16. (*i*) (7125) 10^n for $n \geq 0$

(*ii*) There is no number which reduces by 58 when its leading digit is deleted.

18. 198 **20.** No integral solutions.

21. Solution set: $\{x.10^k \mid x = 10125, 2025, 30375, 405, 50625, 6075, 70875: k \geq 0\}$.

EXERCISE 5.2. (p. 183)

1. 1, 5/3. **2.** 6, – 3. **4.** 3/2, – 4/3. **6.** $-\sqrt{3}/2, -4/\sqrt{3}$

7. $\sqrt{5}/\sqrt{7}, \sqrt{7}/\sqrt{5}$ **8.** $\sqrt{3}, \sqrt{3}/2$. **9.** – 7, 1/6

10. 4, – 2 **12.** 1/3, – 4.

EXERCISE 5.3. (p. 187)

1. (*a*) $(1/4)(-1 \pm i\sqrt{7})$. (*c*) $-1 \pm i\sqrt{3}$.
(*d*) $(1/4)(-1 \pm \sqrt{5})$. (*e*) $-3 \pm \sqrt{3}$.
(*f*) $(1/4)(-5 \pm i\sqrt{7})$. (*g*) $-2, -1/4$.
(*h*) $(1/2\sqrt{3})(-2 \pm \sqrt{10})$. (*j*) $1, -6$.
2. (*a*) $a \geq -81/4$ (*b*) $|a| \geq 4$
(*c*) $a \leq 2$ (*d*) $a \leq -8$ or $a \geq 0$.
(*e*) For all real numbers a. (*f*) $|a| \leq 2$
3. ₹ 9/- 4. $m, \dfrac{1-m^2}{2m}$
5. $1, -1/3$.

EXERCISE 5.4. (p. 191)

1. (*a*) $\alpha + \beta = -9$, $\alpha\beta = 8$ (*c*) $-3, 4/3$.
(*e*) $-7/6, -1/2$. (*g*) $6, -72$.
(*j*) $-3/2, 1/2$.

2.

	$\alpha^3 + \beta^3$	$\alpha/\beta + \beta/\alpha$	$\alpha^2\beta + \alpha\beta^2$
(*a*)	$-7/2$	$-19/9$	$9/8$
(*b*)	464	30	16
(*c*)	$-135\sqrt{3}/2$	16	$-9\sqrt{3}/2$
(*d*)	$67\sqrt{5}/64$	$-43/24$	$-3\sqrt{5}/8$
(*e*)	-3024	-6	432
(*g*)	-43	$-29/14$	14

3. (*b*) $x^2 - 3\sqrt{2}x + 4$. (*e*) $x^2 - (1.8)x + 0.72$.
(*g*) $x^2 - 4x + 13$. (*j*) $x^2 + 2$.
4. $a = -2$, $c = -4$. 5. The other root is 1, $b = -8$.
6. $x^2 - 2x - 2$. 7. $b^3 + ac(a + c) - 3abc = 0$.
$x = 30$, the roots are $4 \pm i\sqrt{14}$. 10. $x^2 + (31/36)x - (449/216)$.
11. $Q^2 + q^2 - p^p(Q + q) - 2qQ + qP^2 + Qp^2$.

EXERCISE 5.5. (p. 194)

1. (*b*) $\pm(\sqrt{2} + 1), \pm(\sqrt{2} - 1)$. (*d*) $1, \pm i$.
(*f*) $\sqrt{2}$ is a double root. (*h*) 0.
(*j*) $\pm\sqrt{5}, \pm\sqrt{2}$. (*l*) $(1/2)(5 \pm \sqrt{29}), (1/2)(3 \pm \sqrt{13})$.
2. 6, 3 4. 3 5. 6, 3

6. 4, 6; – 6, – 4 **8.** 8, – 1 **9.** Sides are 10 and 8 units.
10. $k \geq 1$ or $k \leq -7/9$ **11.** $a = 9$ or $a = 15$.

EXERCISE 5.6.(p. 199)

1. All reals. **2.** $x > 3$ or $x < 1$. **4.** $6 - 4\sqrt{6} < x < 6 + 4\sqrt{6}$.
6. $x < (1/2)(3 - \sqrt{21})$ or $x > (1/2)(3 + \sqrt{21})$.
7. $-1 - \sqrt{13} < x < -1 + \sqrt{13}$. **8.** All reals. **9.** $-1 \leq x \leq 8/3$.
10. $x \leq 8 - \sqrt{73}$, $8 - \sqrt{57} \leq x \leq 8 + \sqrt{57}$ or $x \geq 8 + \sqrt{73}$.
11. $x = 2$ or $-2 \leq x \leq -1$. **12.** $-4 < x < 1$ or $x > 7$.
13. $-4 \leq x \leq 1/2$ or $3 \leq x \leq 5$. **15.** $-3/8 \leq x \leq -1/21$.
16. $|x| > 3$. **17.** $x < -1$ or $0 < x < 1/2$ or $x > 1$.
18. $x < -\sqrt{2}$ or $0 < x < 1$ or $\sqrt{2} < x < 2$.
19. $x \leq -2$ or $-1 \leq x \leq 1$. **20.** $x \leq -3/2$ or $x \geq 1/2$
21. $-3 - \sqrt{5} \leq x \leq -4$ or $-2 \leq x \leq 0$.

PROBLEMS (Chapter 5). (p. 199)

3. $n = 11$. **9.** $(w + a)\,\bar{u}$ is real where $u^2 = a^2 - b$.
10. u and v are real multiples of $a - c$ where $u = \sqrt{a^2 - 4b}$, $v = \sqrt{c^2 - 4d}$.
11. Any polynomial $ax^2 + bx + c$, where ib/a and c/a are real.
15. $x = 5, 10$.

EXERCISE 6.1. (p. 203)

2. 1° 43′ 8″ **3.** (*a*) $(\pi/9)^c$. (*b*) $(\pi/18)^c$.
4. 3,86,700 km. nearly.

EXERCISE 6.2. (p. 205)

2. (*a*) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$. (*b*) $\{n\pi : n \in \mathbf{Z}\}$.

(*c*) $\left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$. (*d*) $\{n\pi : n \in \mathbf{Z}\}$.

17. $\sin\theta = \sqrt{1 - K^2}$, $\tan\theta = \left(\sqrt{1-K^2}\right)/K$,

$\sec\theta = 1/K$, $\operatorname{cosec}\theta = -1/\left(\sqrt{1-K^2}\right)$,

$\cot\theta = K/\left(\sqrt{1-K^2}\right)$.

18. 9/4.

EXERCISE 6.3. (p. 208)

1. (*a*) 1. (*b*) 1. (*c*) $\cos^4 A$.
2. (*a*) – 1/2. (*b*) – 1/2. (*c*) $1/\sqrt{3}$. (*d*) – 2. (*e*) $\sqrt{3}$.
   (*f*) $-2\sqrt{3}$.

EXERCISE 6.4. (p. 214)

2. (*a*) $f(x) = \sqrt[3]{x}$. (*b*) $f(x) = \begin{cases} x/2 & , x \geq 0 \\ x & , x < 0 \end{cases}$.
   (*c*) Not invertible. (*d*) Not invertible.
   (*e*) $f(x) = \tan^{-1} x$. (*f*) Not invertible.
   (*g*) $f^{-1} = f$.
4. (*a*) $R - \{0\}$. (*b*) $R - \{-1, 0, 1\}$.
   (*c*) $[-3, 3]$. (*d*) $(-\infty, -3) \cup (3, \infty)$.
   (*e*) $\bigcup_{n \in z} (2n\pi, (2n+1)\pi)$. (*f*) $R - \{n\pi / n \in Z\}$.
10. (*a*) n^m (*b*) $n!/(n-m)!$
    (*c*) $2^m - 2$ for $n = 2$;
    $3^m - 3 \cdot 2^m + 3$ for $n = 3$.

EXERCISE 6.5. (p. 223)

23. $\dfrac{4 - p^2 - q^2}{p^2 + q^2}, \dfrac{q^2}{p^2}$.

28. $\cos 27° = \dfrac{\sqrt{5+\sqrt{5}} + \sqrt{3-\sqrt{5}}}{4}$

    $\sin 27° = \dfrac{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}}{4}$

29.

	sine	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
15°	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
75°	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$

EXERCISE 6.7. (p. 233)

22. $x = \pm 1/\sqrt{2}$.
23. $x = \sqrt{3}, -\sqrt{3} - 2$.
24. $x = \sqrt{3}/(2\sqrt{7})$.
25. $x = 0, 1/2$.

EXERCISE 6.8. (p. 239)

The values for θ, in Questions 1 – 12 are as follows.

1. $(2n)\, 180° - 104°\, 18'\, 39''$, $n \in Z$.
2. $n\pi$, $n \in Z$.
3. $k\pi/n$; $2k\pi/n \pm \pi/3n$, $k \in Z$.
4. $(2n + 1)\pi/6$, $n \in Z$.
5. $n\pi$, $n\pi \pm \pi/3$, $n \times 180° \pm 35°15'$, $n \in Z$.
6. $n\pi/2$, $n \in Z$.
7. $n\pi + (-1)^n \frac{3\pi}{8} - \frac{\pi}{8}$, $n \in Z$.
8. $(4n + 1)\frac{\pi}{2}$, $(4n + 1)\frac{\pi}{2} - \alpha$, $n \in Z$, where $\alpha = 33°\, 41'\, 24''$.
9. $n\pi + \frac{\pi}{6} \pm \frac{\pi}{5}$, $n \in Z$.
10. $\frac{n\pi}{3} + \frac{\pi}{12}$, $n \in Z$.
11. $n\pi \pm \frac{\pi}{4}$, $n\pi \pm \alpha$, $n \in Z$, where $\alpha = 73°\, 13'\, 17''$.
12. $\frac{n\pi}{2}$, $2n\pi \pm \frac{2\pi}{3}$, $n \in Z$.
13. $(d - a)(d - c) \le b^2$.
14. $\theta = 2n\pi + \frac{\pi}{2}$, $2n\pi + \frac{\pi}{4}$, $n \in Z$.
16. $\cos\theta = \cot A + \cot B + \cot C$.
17. Division by $\cos\theta$ is not allowed if $\cos\theta = 0$.
19. (*a*) $-\sqrt{2}, \sqrt{2}$ (*b*) $-2, 2$ (*c*) $-3, 7$.

    (*d*) $\frac{1}{2}\left[(a+c) - \sqrt{(a-c)^2 + b^2}\right], \frac{1}{2}\left[(a+c) + \sqrt{(a-c)^2 + b^2}\right]$.

    (*e*) $-\sqrt{1 + \sin^2\alpha}$; $\sqrt{1 + \sin^2\alpha}$.
20. $x = n\pi + \frac{\pi}{3}$, $y = n\pi + \frac{\pi}{6}$, $n \in Z$.

EXERCISE 6.9B. (p. 246)

3. $(m - n)\, AD^2 = m \cdot AC^2 - n \cdot AB^2 + \frac{mn}{(m - n)} \cdot BC^2$.

5. $h_a = 48/13, \quad h_b = 12, \quad h_c = 16/5.$

15. 50 m and 21 m

17. (*i*) $\dfrac{na^2}{4} \cot \dfrac{\pi}{n}$ (*ii*) $nr^2 \tan \dfrac{\pi}{n}$ (*iii*) $\dfrac{nR^2}{2} \sin \dfrac{2\pi}{n}$

20. (*i*) $A = 60° 42'$, $B = 42° 39'$, $C = 76° 39'$

(*ii*) $B = 64° 6'$, $C = 39° 54'$, $a = 116.5$

(*iii*) $C = 77°$, $b = 37.36$, $c = 59.13$

(*iv*) (*a*) $C = 63° 27'$, $A = 64° 33'$, $a = 42.40$;

(*b*) $C = 116°33'$, $A = 11° 27'$, $a = 9.321$.

(*v*) $B = 35° 10'$, $C = 104° 50'$, $c = 168.4$.

(*vi*) No solution exists

(*vii*) $C = 90°$, $A = 15°$, $a = 2 - \sqrt{3}$

(*vii*) $B = 37° 48'$, $C = 12° 12'$, $c = 27.59$.

(*ix*) No solution exists

(*x*) $A = 21° 47'$, $B = 38° 13'$, $C = 120°$.

EXERCISE 6.10. (p. 261)

2. $\dfrac{4a(a+b)\sin^2 \alpha + b^2}{2(a+b)\cos \alpha}$

3. If $\alpha = 1 = \beta$, height of tower $= (a \tan \alpha)/\sqrt{3}$. Otherwise it is equal to

$$a \times \sqrt{\frac{p^2 + q^2 + r^2 - \sqrt{2p^2q^2 + 2p^2r^2 + 2q^2r^2 - p^4 - q^4 - r^4}}{2(p^4 + q^4 + r^4 - p^2q^2 - p^2r^2 - q^2r^2)}},$$

where $p = \cot \alpha$, $q = \cot \beta$ and $r = \cot \gamma$.

EXERCISE 6.11. (P. 264)

1. $a^2 + b^2 = c^2 + d^2$.

3. $(x + y)^{2/3} + (x - y)^{2/3} = 2$.

5. $(\lambda^2 - 1)^2 = 27 \lambda^2 \sin^2 \alpha \cdot \cos^2 \alpha$.

7. $(a^2 + b^2 - 1)^2 = 4[(a - 1)^2 + b^2]$.

9. $(b^2 + 1)^2 + 2a(b^2 + 1)(a + b) = 4(a + b)^2$.

11. $(x^2 + y^2 - 1)^2 = (y + 1)^2 + x^2$.

13. $(x + y)^{2/5} + (x - y)^{2/5} = 2$.

15. $m^2 + m \cos \alpha = 2$.

17. $xy = (y - x) \tan \alpha$.

19. $\dfrac{x^2}{p} + \dfrac{y^2}{q} = \dfrac{1}{p} + \dfrac{1}{q}$.

20. $2x^3 + z = 3x(1 + y)$.

PROBLEMS (Chapter 6) (p. 265)

12. $x = 2k\pi$ or $(2k - (1/2))\,\pi$, $k \in Z$, if n is odd; $x = k\pi$, $k \in Z$, if n is even.

13. $x = (2k + 1)\pi/4$, $(2k + 1)\pi/6$, $k \in Z$

14. $[\pi/4, 7\pi/4]$

25. Angles : $\pi - 2A$, $\pi - 2B$, $\pi - 2C$;

Sides : $a\,\lambda \cos A$, $b\,\lambda \cos B$, $c\,\lambda \cos C$;

Area : $2\Delta\lambda^2 \cos A \cos B \cos C$;

Circumradius : $\frac{1}{2}R\lambda$;

Inradius : $R(1 + \cos A \cos B \cos C)$,

where $\lambda = (1 + \cos A + \cos B + \cos C)/2 \cos A \cos B \cos C$

27. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = 0$

EXERCISE 7.2. (p. 301)

3. $y = (3/2)x$, $y = (1/6)x$

5. $y = 0$

9. $(c_2 - c_1)^2/(m_1 - m_2)$

11. $y + 2x = 7$

18. (*i*) $x - 3y - 8 = 0$; (*ii*) $3x - 16y - 30 = 0$

20. $\alpha = 15°$ or $75°$

EXERCISE 7.3. (p. 313)

1. $3x^2 + 3y^2 + 2x - 20y + 17 = 0$

2. $25x^2 + 25y^2 - 100x - 150y + 156 = 0$

9. $x^2 + y^2 = 0$.

17. $x^2 + y^2 + 3x + 3y = 0$

18. $4ax - y^2 = 3a^2$

PROBLEMS (Chapter 7). (p. 314)

14. $a(1 + \cos\alpha) + p = 0$. or $a(1 - \cos\alpha) + p = 0$

33. 32 square units.

43. $c^2x^2 - y^2(a^2 - b^2 + abc) - 2bcxy - ac^2x + a^2yc = 0$,

where $(a, 0)$ is A and (b, c) is B

EXERCISE 8.1. (p. 327)

1. $x = 3$, $y = -2$

2. No solution.

3. $x = 5 + y$, y arbitrary

4. $x = 5$, $y = -1$

5. $x = 1 + y$, $z = 2 - 2y$, y arbitrary.

6. $x = -(11/3) + (13/3)w + (2/3)u$, $y = 5 - 2w - u$,

$z = (7/3) - (8/3)w - (1/3)u$, w and u being arbitrary.

7. $x = 2, y = -1, z = -1$
8. $x = 13/9, y = 5/3, z = -5$
9. $x = 2 - y, z = -1$, y arbitrary
10. $x = 2, y = -1 - z$, z arbitrary
11. $x = 4, y = 3, z = 5$
12. No solution.
13. $x = -1, y = -1, z = -2$.
14. $x = -3z + 2w - 3, y = 5 - w$, w and z being arbitrary.
15. $x = -w + 14, y = z + 4$, w and z arbitrary.
16. $x = (-6/4)z + (14/4), y = (6/4)z - (10/4)$.
17. $x = -2, y = 3, z = 3/2, w = -1/2$.

EXERCISE 8.2. (p. 350)

1. (a) 17 (b) – 45 (c) 25 (d) – 4
2. $A_{11} = -5$, $A_{12} = 10$, $A_{13} = -5$,
   $A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$,
   $A_{31} = 4$, $A_{32} = -8$, $A_{33} = 4$,
   $\det A = 0$
3. $A_{11} = 16$, $A_{12} = -64$, $A_{13} = -64$
   $A_{14} = 48$, $\det A = -192$
4. $(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$
5. $\lambda = 0, \dfrac{3 \pm \sqrt{33}}{2}$

PROBLEMS. (Chapter 8) (p. 351)

1. (a) $x = 0, y = 5/9, z = -5/9, w = -1$
   (b) No solution.
2. If $k^2 \neq 1$, the system has one solution.
   If $k = 1$, it has infinitely many solutions.
   If $k = -1$, it has no solution.
7. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
9. $= 0$

EXERCISE 9.1. (p. 381)

1. 2160; 720; 144
3. 7^{10} ; $7! \times 7^3$
5. 2^6
7. $\binom{20}{5} 2^5$
9. 4536; 2296; 120
15. $\binom{n+k}{k}$
17. $(n - 1)!$
19. $\dfrac{8!}{(2!)^4}$

20. $\binom{n}{m}$ 22. 2^n ; $\binom{n}{m}$ 24. $6! \times \binom{7}{2}$; $5! \times \binom{5}{2}$

EXERCISE 9.3. (p. 387)

3. (*a*) Zero (*b*) No term independent of x.
   (*c*) -672×3^6
5. 3

PROBLEMS (Chapter 9). (p. 387)

1. $\frac{n(n-1)}{4}$ 2. (*a*) 240 (*b*) 120
3. $2^{10} - 2^5$ 6. $\frac{n(n+1)\ldots(n+r-1)}{r!}$; $\binom{n}{r}$
13. 11!; 5! 6! 15. $9[1 + 9_1 + 9_2 + \ldots + 9_9]$
16. $\sum_{k=j}^{m} P(m, k)\binom{n}{k}$ where $P(m, k)$ is the number of ordered partitions of m into k parts.

EXERCISE 10.1. (p. 392)

1. $x^6 + 8x^4 + 9x^3 + 10x^2 + 12x + 13$;
   $8x^{10} + 9x^9 + 12x^7 + 84x^6 + 90x^5 + 72x^4 + 201x^3 + 40x^2 + 108x + 36$
3. $x^9 + x^8 + x^7 + \ldots + x^2 + 2x$; $x^{10} - 1$
5. $\sqrt{7}x^4 + (8+\sqrt{7})x^3 - 6x^2 + 12x + 4$;
   $7x^7 + 2\sqrt{7}x^6 + (2\sqrt{7} - 48)x^5 + 16x^4 + 4\sqrt{7}x^3 - 24x^2 + 8x$
7. $7x^6 + (1/2)x^4 + 3x^3 + 3x^2/4 + 35/6$
   $\frac{7}{2}x^{10} + \frac{21}{4}x^8 + \frac{3}{2}x^7 + \frac{35}{6}x^6 + \frac{9}{4}x^5 + \frac{5}{2}x^4 + \frac{15}{6}x^3 + \frac{15}{4}x^2 + \frac{25}{6}$.
9. $x^5 - x^4 + x^3 - x^2 + 2x$; $x^6 - 1$
11. $2\sqrt{2}x^2$; $-3x^6 + (2 - 2\sqrt{3})x^4 - x^2$
13. $10x^3 + 10x^2 + 3x + 4$;
    $10x^6 + 19x^5 + 31x^4 + 41x^3 + 23x^2 + 6x + 2$

EXERCISE 10.2. (p. 397)

In the answers to Questions 1 and 2 below, the first mentioned expression is $q(x)$ and the second mentioned is $r(x)$.

1. (*b*) $x^5 + x - 1$; $x^3 + x^2 - x + 2$.
   (*d*) $x^5 + 2^{1/6}x^4 + 2^{1/3}x^3 + 2^{1/2}x^2 + 4^{1/3}x + 2^{5/6}$; 0
   (*f*) $4x^2 + 6x$; $11x + 9$

(*h*) $x - 7; 9x + 1$

(*j*) $- 20x^2 + 7x + 13; - 6x + 6$

(*r*) $x^3 + 2x^2 + 5x + 10; 21x + 21$

2. (*a*) $2x^2 - 11x + 15; 0$

(*c*) $5x^2 - 27x; 18$

(*e*) $x^9 + 2x^8 + 4x^7 - x^6 - 2x^5 - 4x^4 - 5x^3 - 10x^2 - 20x - 40; - 76$

(*g*) $x^4 + 2x^2 + 3; 14$

(*i*) $x^2 - 3x + 18; 112$

(*k*) $4x^2 + 4x; 3x^2 + 4x + 2$

(*m*) $x^3 - 2x^2 - 6x + 12; - 15$

(*o*) $x^3 + 4x^2 - 14x + 9; - 15x - 2$

3. $(x - 3)^3$

6. $u^{10} + 10u^9 + 45u^8 + 120u^7 + 210u^6$
$+ 252u^5 + 210u^4 + 120u^3 + 45u^2 + 10u$, where $u = x - 1$

8. $u^4 + 3u^3 + 9u^2 + 27u + 17$ where $u = x - 3$

10. $(u^6 + 6u^5 + 15u^4 + 29u^3 + 42u^2 + 65u + 58)$
$\times (u^5 + 5u^4 + 10u^3 + 10u^2 - 2u)$ where $u = x - 1$

EXERCISE 10.3. (p. 404)

1. 'No' Answers: (*a*), (*b*), (*d*), (*l*), (*m*).
The remaining answers are 'yes'.

EXERCISE 10.4. (p. 406)

1. In each of the answers below, the first – mentioned is *the gcd* and the second is *the lcm*.

(*a*) $(x^2 + 2)(x - 1)$; $(x^2 + 2)(x + 9)(x^5 - 1)(x^4 + 2)$

(*b*) $\left(x + \frac{1}{2}\right)$; $\left(x^2 - \frac{16}{9}\right)\left(x + \frac{1}{2}\right)(x^2 - 3x + 2)(x + 4)$

(*c*) $(x - 3)(x - \sqrt{2})$; $\left(x - \frac{3}{\sqrt{2}}\right)(x^2 - 9)(x^2 - 2)$

(*d*) $(x + 1)\left(4x + \frac{1}{4}\right)$;

$(x^3 - 12x^2 + 47x - 60) \times \left(x^2 - \frac{5}{12}x - \frac{1}{6}\right) \times \left(x^2 - \frac{5}{9}x - \frac{4}{9}\right)$

(*e*) $x^3 - x^2 + x - 1; x^4 - 1$

(*f*) $(x + \sqrt{2})(x^2 + 1)(x - 2)$; $x(x + \sqrt{2})(x^2 + 1)(x - 2)^2$

(*g*) $(x + 3)\left(x + \frac{1}{\sqrt{3}}\right)$; $(x + 1)(x + 3)\left(x + \frac{1}{\sqrt{3}}\right)^2$

(*h*) $x(x + 4)\left(x + \frac{1}{\sqrt{3}}\right)$; $x^3\left(x^2 + \frac{4}{\sqrt{3}}x + 1\right) \times (x^2 + 4x + 3)(x + \sqrt{5})$

EXERCISE 10.5. (p. 413)

2. (*a*) $(x + 1)(x + 3)$; $(x + 1)(x^2 + 2)$
   (*b*) $(x^2+1)(x^4+1)$; $(x^4 - 1)$
   (*c*) $(x + 1)(x^3 + 2)$; $(x + 1)(x^2 - x + 1)^2$
   (*d*) $(x + 1)^2 (x - 1)$; $(x + 1)^2 (x + 2)$

PROBLEMS (Chapter 10). (p. 414)

4. $l(x) = \left(x^2 + \frac{9}{4}x + \frac{3}{2}\right)$, $m(x) = \left(x^2 - \frac{7}{4}x + \frac{1}{2}\right)$.

11. $6x^5 - 15x^4 + 10x^3$

PROBLEMS (Chapter 11). (p. 433)

15. $x = 3/4$, $y = 1/2$, $z = 1/4$ 17. $2\sqrt{2}$

18. The expression has constant value equal to 2.

19. No 20. $(a + k)^n$

EXERCISE 12.1 (p. 444)

1. 582 2. 1806
3. 312 7. 66, 40, 02,144.

PROBLEMS (Chapter 12). (p. 454)

1. $10^n - 4.9^n + 6.8^n - 4.7^n + 6^n$
2. 120.
5. $D_{n/2}$ or $D_{(n-1)/2}$ according as n is even or odd.

PROBLEMS (Chapter 13). (p. 470)

4. (*a*) $\left[\binom{10}{3}\binom{8}{4} + \binom{10}{2}\binom{8}{5} + \binom{10}{1}\binom{8}{6} + \binom{8}{7}\right] \div \binom{8}{7}$

   (*b*) $1 \div \dfrac{\binom{8}{7} + \binom{10}{7}}{\binom{18}{7}}$

5. 0.665 6. 1/4 7. 0 8. No 9. 21/100

10. $1 - \dfrac{2^n}{3^n}$ 11. $\dfrac{4}{\binom{52}{13}}$

EXERCISE 15.1. (p. 487)

1. 1/6, 3/6, 9/6, 27/6, 81/6
3. All constant sequences.
4. 1, 2, 4, 8, 16, 32, 64, 128, 256
5. 4, 12, 36
6. 3, – 6, 12, – 24
7. – 5, – 13, – 21, – 29
8. – 6, – 3, 0, 3, ..., 42
11. 32

EXERCISE 15.2. (p. 490)

2. 275
3. 0
4. 20, 18, 16, 14, ..., – 18
6. $n = 17$, $u_n = -24$
7. 9119700
8. 38
10. 4905
12. 55
13. 2, 4, 6, 8, ..., 20
14. 8825

EXERCISE 15.3. (p. 492)

1. 5, 15, 45, ..., 3645
5. 3, 7, 11 or 12, 7, 2
6. 3 or – 3

EXERCISE 15.4. (p. 496)

1. $\dfrac{n(n+1)(2n+13)}{6}$
3. $\dfrac{n+1}{n+2}$
4. $\dfrac{n}{2n+1}$
5. $\dfrac{n(n+3)}{4(n+1)(n+2)}$
6. $\dfrac{8n+9}{12(2n+3)}$
15. $\dfrac{2n^2+2n+1}{4(2n+1)(2n+3)}$
18. $\dfrac{5(n-4)}{24(n+2)}$
23. 2926
27. $\sqrt{n+1}-1$

EXERCISE 15.6. (p. 504)

1. $n(n-1)\,2^{n-2}$
3. $\dfrac{2^{n+1}-1}{n+1}+2^n$
8. $n \cdot 2^{n-1}$
12. $\dfrac{n(n+1)}{2}$
16. $\dfrac{2^n-1}{n!}$
17. $\dfrac{n(n+1)}{30}(6n^3+9n^2+n-1)$

PROBLEMS (Chapter 15) (p. 506)

5. $\dfrac{(n-1)n\,(n+1)\,(3n+2)}{24}$

7. $\dfrac{n(n+1)}{4(n+2)}$

EXERCISE 16.1. (p. 512)

1. $\cos 14\alpha + i \sin 14\alpha$

3. modulus $= \left(\dfrac{1+\sin\theta+\cos\theta}{1+\cos\theta}\right)^3 . 2^{-(3/2)}$;

amplitude $= \pi/4$

7. (a) $\dfrac{\sin(n\,\theta/2)}{\sin(\theta/2)}\left(\sin\left(\dfrac{n+1}{2}\right)\theta\right)$

(b) $\dfrac{n}{2}+\dfrac{1}{2}\,\dfrac{\sin n\,\theta\cos(n+1)\,\theta}{\sin\theta}$

EXERCISE 16.2. (p. 515)

1. (a) $2^{1/6}\,\text{cis}\left(\dfrac{\pi}{9}+\dfrac{2k\pi}{6}\right), k = 0, 1, ..., 5$

(b) $2^{1/6}\,\text{cis}\left(\dfrac{\pi}{12}+\dfrac{2k\pi}{3}\right), k = 0, 1, 2$

5. $6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

6. $\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

PROBLEMS (Chapter 16). (p. 516)

3. (a) $\tan(2^n\theta) - \tan\theta$

5. $16x^4 - 8x^3 - 12x^2 + 4x + 1 = 0$

6. $x = y = z = \dfrac{a+b+c}{3}$

MISCELLANEOUS PROBLEMS (P. 518)

9. (a) $(bc + ca + ab)(bc + ca + ab - a^2 - b^2 - c^2)$

(b) $(a^2 + b^2 + c^2 - bc - ca - ab)$
$\times [(a^2 + b^2 + c^2 - ab - bc - ca)(ab + bc + ca)^2$
$- 2(a^2 - bc)(b^2 - ca)(c^2 - ab)]$

45. (a) $a^4 + b^4 = c^2(a^2 + b^2)$

(b) $(bc + cd + ab)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

(c) $a^2 + b^2 = 6ab$

(d) $(p + q)^{2/3} + (p - q)^{2/3} = 2$

(e) $(a + b)^{2/3} + (a - b)^{2/3} = 4c^{2/3}$

(*f*) $c^2 (a + b - 1)^2 - c(a + b - 1) (a^2 - 2ab + b^2 - a - b) + ab = 0$

(*g*) $a^2 = c^2 \pm bc$ or $a^2 = b^2 - c^2$

(*h*) $a^3 + 2c^3 = 3ab^2$

59. (*a*) $a^3 + b^3 + c^3 = a^2 (b + c) + b^2 (c + a) + c^2 (a + b)$

(*b*) $a^3 - 3ab + 2c - bd = 0$

(*c*) $(a^3 - 3ab^2 + 2c^3) (a^4 - 3a^2b^2 + 3b^4 - 4ac^3) + 16d^2 (a^2 - b^2)^2 = 0$

(*d*) $a^2 + b^2 + c^2 - 2abc - 1 = 0$

(*e*) $a^3 + b^3 + c^3 - 3abc = 2a^2$

(*f*) $ab = c + 1$

(*g*) $(a^2 - 2b) (a^2 + b)^2 = 2(a^3 - 2ab + 2c)^2$

(*h*) $2d (1 - bc - ca - ab) = (1 - d^2) (a + b + c - abc)$

(*i*) $a^2 + b^2 + c^2 - abc - 4 = 0$

(*j*) $2(b^2 c^2 + c^2a^2 + a^2b^2 - a^4 - b^4 - c^4) + a^2b^2c^2 = 0$

109. (*a*) $a^3 + b^3 + c^3 = a^2(b + c) + b^2(c + a) + c^2(a + b)$

(*b*) $abc = (a + b + c - 4)^2$

(*c*) $A^2 + B^2 + C^2 + 3(BC + CA + AB) = 0$, where

$A = a^2 - bc, B = b^2 - ac, C = c^2 - ab$

(*d*) $(a^2 + b^2) = (\lambda^4 - 4\lambda^2 + 2)ab$

(*e*) $(a + b + c)^3 - 4(b + c) (c + a) (a + b) + 5abc = 0$

(*f*) $$\frac{1}{(a-b)cr+(a-c)bq}+\frac{1}{(b-c)ap+(b-a)cr}$$

$$+\frac{1}{(c-a)bq+(c-b)ap}+\frac{1}{bcqr+carp+abpq}$$

(*g*) $(l + m + n)^2 = 3h^2$

(*h*) $(b^2 - ca)^2 (c^2 - ab)^2 + (c^2 - ab)^2 (a^2 - bc)^2 + (a^2 - bc)^2 (b^2 - ca)^2$
$= d^2(a^3 + b^3 + c^3 - 3abc)^2.$

Index

INDEX

(of technical words & names with page numbers of their first/major appearance/s)

A

A S A Theorem, 43
A.P, 488
Absolute Value, 10
Adjacent Angles, 40
Alternate Angles, 50
AM-GM Inequality, 425
Analytical Geometry, 267
Angle of Depression, 258
Angle of Elevation, 258
Apollonius's Theorem, 246
Argand Diagram, 9
Argument of a Complex Number, 10
Arithmetic Mean, 419
Arithmetic Progression, 66
Aryabhatta, 202
Augmented Matrix, 351

B

Bhaskara, 202
Bijective, 210
Binomial Coefficients, 383
Binomial Theorem, 382
Brahmagupta, 202
Brahmagupta's Theorem, 257
Butterfly Theorem, 169

C

Cartesian Geometry, 267
Cauchy-Schwarz Inequality, 429
Centres of Similitude, 96
Centroid, 59
Ceva's Theorem, 77
Cevian, 79
Chinese Remainder Theorem, 480
Circumcentre, 58
Cofactor, 332
Co-domain, 209
Coaxial System, 307
Collinear Points, 74
Common Difference, 476
Common Ratio, 468
Complete Graph, 447
Complex Number, 8
Composite Function, 212
Concentric Circles, 86
Concyclic Points, 36
Congruence, 36
Congruent Triangles, 42
Constant Function, 216
Convex Polygon, 53
Convex Quadrilateral, 54
Coprime, 23
Corresponding Angles, 38

Cosine Rule, 240
Coterminal Angles, 205
Countability, 8
Cramer's Rule, 344
Cross-multiplication, 326
Cube Roots of Unity, 514
Cyclic Quadrilateral, 99

D

De Moivre's Theorem, 509
Dedekind, 7
Degrees of Freedom, 348
Derangement, 441
Descartes, 267
Determinant, 328
Discriminant of a Quadratic, 183
Divisibility of Numbers, 18
Divisibility of Polynomials, 398
Division Algorithm for Numbers, 25
Division Algorithm for Polynomials, 407
Divisor of a Number, 18
Divisor of a Polynomial, 401
Domain, 289

E

Elementary Events, 459
Eliminant, 263
Equation of curve, 275
Equiangular Triangles, 67
Erdos-Mordell Theorem, 157
Erdos-Szekeres Theorem, 450
Euclid, 20
Euclid's Algorithm for Polynomials, 25
Euclidean Algorithm, 25
Euclidean Distance, 269
Euler, 118
Euler's Theorem, 116, 256
Euler Function, 444
Euler Line, 123
Event, 355
Excentral Triangle, 253
Excentre, 103
Excircles, 80
Exradii, 131
Extension, 3

F

Factor, 5
Factorial *n*, 360
Fermat's Theorem, 478
Fermat Point, 161
Feucrbach's Theorem, 123
Fibonacci Sequence, 18
Finite Real Sequence, 485
Fourth Proportional, 135
Friendship Theorem, 448
Fundamental Principle of Counting, 354
Fundamental Theorem of Algebra, 9
Fundamental Theorem of Arithmetic, 27

G

g.c.d, 20
g.c.d. of Polynomials, 413
G.P., 486
Gauss, 472
Geometric Mean, 419
Geometric Progression, 15
Greatest Common Divisor, 20

H

Harmonic Conjugates, 66
Harmonic Progression, 66
Heron's Formula, 245
Horner's Method, 394

I

Identity Function, 212

Im f, 209
Imaginary Axis, 9
Implicit Function, 213
Incentre, 59, 61, 74
Incircle, 109
Inclusion and Exclusion Principle, 436
Independent Events, 357, 361
Inner Napolean Triangle, 164
Inradius, 113, 115
Inverse Circular Functions, 229, 231
Inverse of a Function, 212
Irrational Number, 7
Irreducible Over **R**, 197
Isosceles Triangle, 44
Isotomic Points, 112
Isotomic Conjugates, 83
l.c.m, 24
l.c.m of Polynomials, 407

L

Laplace Expansion, 333
Leading Coefficient, 198
Least Common Multiple, 24
Linear Algebra, 325
Linear Function, 194
Linear System, 318
Locus, 47
Logarithmic Function, 213

M

m.e., 458
Mathematical Induction. 15, 17
Matrix, 331
Mean Proportional, 73
Medial Triangle, 80
Menelaus's Theorem, 75
Modulus of a Complex Number, 10
Monic Polynomials, 406
Morley's Theorem, 70
Multinomial Coefficient, 365
Mutually Exclusive Events, 361, 458, 459

N

Nine-Point Centre, 123, 152
Nine-Point Circle Theorem, 122
Number π, 202
Number e, 466
Number i, 8

O

One-one Function, 217
One-one Correspondence, 8, 275
Onto Function, 217
Orthic Triangle, 80
Orthocentre, 60, 84
Outer Napolean Triangle, 164

P

P H P, 445
Parallel Lines, 52
Parallelogram, 54
Pascal's Triangle, 386
Pedal Line Theorem, 127
Pedal triangle. 119, 127
Pedal Triangle of a Point, 127
Perfect number, 482
Pigeon-Hole Principle, 445
Polynomial, 13
Polynomials in Two Variables, 415
Power of a Point w.r.t. a Circle, 303
Prime Number, 23
Ptolemy's Theorem, 102, 103, 128
Pythagoras's Theorem, 70, 107

Q

Quadratic Equations, 177
Quadratic Expression, 13

R

R H S Theorem, 46
Radian, 175
Ramsey's Theorem, 452
Rational Numbers, 4
Real Axis, 9
Real Number System, 8
Reciprocal Transversal, 83
Redundant Combinations, 379
Relatively Prime, 23
Remainder Theorem, 398
Right Angle, 47
Roots of a Quadratic Equation, 183
Row Rank of Matrix, 352

S

S A S Test, 42
S S S Theorem, 46
Sample Space, 456
Section Formula, 271
Set C, 9
Set N, 1
Set Q, 4
Set R, 5
Set Z, 4
Shoemaker's Knife, 173
Sieve Formula, 439
Similar Triangles, 61
Simson Line, 127
Sine rule, 240
Size of Matrix, 331
Square Matrix, 331
Stewart's Theorem, 74
Sum-Free Set, 450
Synthetic Division, 394

T

Tangent, 91, 93
Telescoping Sum Technique, 495
Transpose of Matrix, 334
Trigonometry, 202

U

Uncountable, 8
Unfriendly Set, 389
Unique Factorisation, 28

V

Vandermonde's Identity, 377
Varahamihira, 202
Venn Diagram, 209
Vertically Opposite Angles, 41
Vieta, 202

W

Weierslrass Inequality, 420
Wilson's Theorem, 479

Z

Zero of a Function, 182
Zero Polynomial, 390
n^{th} Roots of a Complex Number, 513
k_n, 448
${}^nP_r$, $p(n, r)$, n_r, 359
$[x]$, integral part of x, 28
$\left(\frac{n}{r}\right)$, n_{c_r}, 369
a/b, 19
(a, b), 21
$[a, b]$, 25
$[z]$, 10
$\tau(n)$, 29
$\sigma(n)$, 29
h_a, h_b, h_c, 115
t_a, t_b, t_c, 145
r_a, r_b, r_c, 249
r_1, r_2, r_3, 110, 249
m_a, m_b, m_c, 145